数と式 5　根号を含む式の計算

分母の有理化（$a>0, b>0, a\neq b$ のとき）

$$\frac{A}{\sqrt{a}}=\frac{A}{\sqrt{a}}\times\frac{\sqrt{a}}{\sqrt{a}}=\frac{A\sqrt{a}}{\sqrt{a}\times\sqrt{a}}=\frac{A\sqrt{a}}{\left(\sqrt{a}\right)^2}=\frac{A\sqrt{a}}{a}$$

$$\frac{A}{\sqrt{a}+\sqrt{b}}=\frac{A}{\sqrt{a}+\sqrt{b}}\times\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{A\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)}=\frac{A\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2}=\frac{A\left(\sqrt{a}-\sqrt{b}\right)}{a-b}$$

$$\frac{A}{\sqrt{a}-\sqrt{b}}=\frac{A}{\sqrt{a}-\sqrt{b}}\times\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{A\left(\sqrt{a}+\sqrt{b}\right)}{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)}=\frac{A\left(\sqrt{a}+\sqrt{b}\right)}{\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2}=\frac{A\left(\sqrt{a}+\sqrt{b}\right)}{a-b}$$

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(1)

$$\begin{aligned}\left(1+\sqrt{2}-\sqrt{3}\right)^2&=\left\{1+\sqrt{2}+\left(-\sqrt{3}\right)\right\}^2\\&=1^2+\left(\sqrt{2}\right)^2+\left(-\sqrt{3}\right)^2+2\cdot1\cdot\sqrt{2}+2\cdot\sqrt{2}\cdot\left(-\sqrt{3}\right)+2\cdot\left(-\sqrt{3}\right)\cdot1\\&=1+2+3+2\sqrt{2}-2\sqrt{6}-2\sqrt{3}\\&=6+2\sqrt{2}-2\sqrt{3}-2\sqrt{6}\end{aligned}$$

(2)

$$\begin{aligned}\left(3-\sqrt{2}-\sqrt{11}\right)\left(3-\sqrt{2}+\sqrt{11}\right)&=\left\{\left(3-\sqrt{2}\right)-\sqrt{11}\right\}\left\{\left(3-\sqrt{2}\right)+\sqrt{11}\right\}\\&=\left(3-\sqrt{2}\right)^2-\left(\sqrt{11}\right)^2\\&=9-6\sqrt{2}+2-11\\&=-6\sqrt{2}\end{aligned}$$

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(1)

$$\begin{aligned}\frac{3\sqrt{5}-5\sqrt{3}}{\sqrt{5}+\sqrt{3}}+\frac{3\sqrt{5}+4\sqrt{3}}{3\sqrt{5}-4\sqrt{3}}&=\frac{3\sqrt{5}-5\sqrt{3}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{3\sqrt{5}+4\sqrt{3}}{3\sqrt{5}-4\sqrt{3}}\times\frac{3\sqrt{5}+4\sqrt{3}}{3\sqrt{5}+4\sqrt{3}}\\&=\frac{\left(3\sqrt{5}-5\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)}+\frac{\left(3\sqrt{5}+4\sqrt{3}\right)^2}{\left(3\sqrt{5}-4\sqrt{3}\right)\left(3\sqrt{5}+4\sqrt{3}\right)}\\&=\frac{15-3\sqrt{15}-5\sqrt{15}+15}{5-3}+\frac{45+24\sqrt{15}+48}{45-48}\\&=\frac{30-8\sqrt{15}}{2}+\frac{93+24\sqrt{15}}{-3}\\&=15-4\sqrt{15}-31-8\sqrt{15}\\&=-15-12\sqrt{15}\end{aligned}$$

(2)

$$\frac{\sqrt{2}-1}{\sqrt{2}+1}+\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}+\frac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}}=\frac{\sqrt{2}-1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1}+\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\frac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$=\frac{\left(\sqrt{2}-1\right)^2}{2-1}+\frac{\left(\sqrt{3}-\sqrt{2}\right)^2}{3-2}+\frac{\left(\sqrt{3}+\sqrt{2}\right)\left(2+\sqrt{3}\right)}{4-3}$$

$$=\left(\sqrt{2}-1\right)^2+\left(\sqrt{3}-\sqrt{2}\right)^2+\left(\sqrt{3}+\sqrt{2}\right)\left(2+\sqrt{3}\right)$$

$$=3-2\sqrt{2}+5-2\sqrt{6}+2\sqrt{3}+3+2\sqrt{2}+\sqrt{6}$$

$$=11+2\sqrt{3}-\sqrt{6}$$

55

(1)

$$x+y=\frac{\sqrt{5}+2}{\sqrt{5}-2}+\frac{\sqrt{5}-2}{\sqrt{5}+2}=\frac{\sqrt{5}+2}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2}+\frac{\sqrt{5}-2}{\sqrt{5}+2}\cdot\frac{\sqrt{5}-2}{\sqrt{5}-2}$$

$$=\left(\sqrt{5}+2\right)^2+\left(\sqrt{5}-2\right)^2$$

$$=9+4\sqrt{5}+9-4\sqrt{5}$$

$$=18$$

(2)

$$xy=\frac{\sqrt{5}+2}{\sqrt{5}-2}\cdot\frac{\sqrt{5}-2}{\sqrt{5}+2}$$

$$=1$$

(3)

$$x^2y+xy^2=xy(x+y)$$

$$=1\cdot 18$$

$$=18$$

(4)

$$x^2+y^2=(x+y)^2-2xy$$

$$=18^2-2\cdot 1$$

$$=322$$

(5)

$$x^3+y^3=(x+y)^3-3x^2y-3xy^2$$

$$=(x+y)^3-3xy(x+y)$$

$$=18^3-3\cdot 1\cdot 18$$

$$=5778$$

あるいは

$$x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$$

$$=18(322-1)\quad\left(\because x^2+y^2=322,xy=1\right)$$

$$=5778$$

重要：2 変数の対称式とその基本対称式

x^2+y^2， x^3+y^3， x^2y^2， $\frac{1}{x}+\frac{1}{y}$， $x+y$， xy など，

x の値と y の値を入れ替えても同じ値になるような式を対称式という。

また， $x+y$， xy を基本対称式といい，対称式はすべて基本対称式で表すことができる。

$x^2+y^2=\left(x^2+2xy+y^2\right)-2xy=(x+y)^2-2xy$

$x^3+y^3=\left(x^3+3x^2y+3xy^2+y^3\right)-3x^2y-3xy^2=(x+y)^3-3xy(x+y)$

$x^3+y^3=(x+y)\left(x^2-xy+y^2\right)=(x+y)\left\{(x+y)^2-3xy\right\}$

ちなみに， $x^3+y^3+z^3$， $x^2y^2z^2$ など 3 変数の対称式の基本対称式は

$x+y+z$， $xy+yz+zx$， xyz の 3 つである。

56

(1)

$$
\begin{aligned}
x+\frac{1}{x}&=\sqrt{2}-1+\frac{1}{\sqrt{2}-1}\\
&=\sqrt{2}-1+\frac{1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}\\
&=\sqrt{2}-1+\sqrt{2}+1\\
&=2\sqrt{2}
\end{aligned}
$$

(2)

$$
\begin{aligned}
x^2+\frac{1}{x^2}&=\left(x+\frac{1}{x}\right)^2-2\\
&=\left(2\sqrt{2}\right)^2-2\\
&=6
\end{aligned}
$$

(3)

$$
\begin{aligned}
x^3+\frac{1}{x^3}&=\left(x+\frac{1}{x}\right)^3-3x\cdot\frac{1}{x}\left(x+\frac{1}{x}\right)\\
&=\left(2\sqrt{2}\right)^3-3\cdot2\sqrt{2}\\
&=10\sqrt{2}
\end{aligned}
$$

(4)

$$
\begin{aligned}
x^5+\frac{1}{x^5}&=\left(x^2+\frac{1}{x^2}\right)\left(x^3+\frac{1}{x^3}\right)-\left(x+\frac{1}{x}\right)\\
&=6\cdot10\sqrt{2}-2\sqrt{2}\\
&=58\sqrt{2}
\end{aligned}
$$

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(1)

$$\frac{10}{\sqrt{3}+\sqrt{2}}=\frac{10\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}$$
$$=10\left(\sqrt{3}-\sqrt{2}\right)$$
$$=10(1.7321-1.4142)$$
$$=10\cdot 0.3179$$
$$=3.179$$

(2)

$$\frac{1}{\sqrt{12}-\sqrt{2}}=\frac{\sqrt{12}+\sqrt{2}}{12-2}$$
$$=\frac{2\sqrt{3}+\sqrt{2}}{10}$$
$$=\frac{2\cdot 1.7321+1.4142}{10}$$
$$=\frac{4.8784}{10}$$
$$=0.48784$$

58

(1)

無理数を分離する。つまり $x=1-\sqrt{5}$ を $x-1=-\sqrt{5}$ とする。

すると，

$$x^2-2x-4=(x-1)^2-5$$
$$=\left(-\sqrt{5}\right)^2-5$$
$$=0$$

(2)

$x^2-2x-4=0$ より， $x^2-2x=4$

よって，

$$x^3-2x^2=x\left(x^2-2x\right)$$
$$=\left(1-\sqrt{5}\right)\cdot 4$$
$$=4-4\sqrt{5}$$

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(1)

$$\frac{\sqrt{2}}{\sqrt{2}-1}=\frac{\sqrt{2}\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}=2+\sqrt{2}$$

$1<\sqrt{2}<2$ より， $2+1<2+\sqrt{2}<1+2$

よって， $a=3$

(2)

$2+\sqrt{2}=a+b$， $a=3$ より， $b=-1+\sqrt{2}$

(3)

$$\begin{aligned} a+b+b^2 &= a+b(1+b) \\ &= 3+\left(-1+\sqrt{2}\right)\left\{1+\left(-1+\sqrt{2}\right)\right\} \\ &= 3+\left(-1+\sqrt{2}\right)\cdot\sqrt{2} \\ &= 5-\sqrt{2} \end{aligned}$$

あるいは,

$$\begin{aligned} a+b+b^2 &= (a+b)+b^2 \\ &= 2+\sqrt{2}+\left(-1+\sqrt{2}\right)^2 \\ &= 2+\sqrt{2}+\left(3-2\sqrt{2}\right) \\ &= 5-\sqrt{2} \end{aligned}$$

60

(1)

$$\begin{aligned} \frac{1}{1+\sqrt{2}-\sqrt{3}} &= \frac{1}{\left(1+\sqrt{2}\right)-\sqrt{3}}\cdot\frac{\left(1+\sqrt{2}\right)+\sqrt{3}}{\left(1+\sqrt{2}\right)+\sqrt{3}} \\ &= \frac{1+\sqrt{2}+\sqrt{3}}{\left(1+\sqrt{2}\right)^2-\left(\sqrt{3}\right)^2} \\ &= \frac{1+\sqrt{2}+\sqrt{3}}{\left(3+2\sqrt{2}\right)-3} \\ &= \frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}} \\ &= \frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}+2+\sqrt{6}}{4} \\ &= \frac{2+\sqrt{2}+\sqrt{6}}{4} \end{aligned}$$

(2)

$$\frac{\sqrt{5}+\sqrt{3}+\sqrt{2}}{\sqrt{5}+\sqrt{3}-\sqrt{2}}=\frac{\sqrt{5}+\sqrt{3}+\sqrt{2}}{\sqrt{5}+\left(\sqrt{3}-\sqrt{2}\right)}\cdot\frac{\sqrt{5}-\left(\sqrt{3}-\sqrt{2}\right)}{\sqrt{5}-\left(\sqrt{3}-\sqrt{2}\right)}$$

$$=\frac{\left(\sqrt{5}+\sqrt{3}+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{3}+\sqrt{2}\right)}{5-\left(\sqrt{3}-\sqrt{2}\right)^2}$$

$$=\frac{\left\{\left(\sqrt{5}+\sqrt{2}\right)+\sqrt{3}\right\}\left\{\left(\sqrt{5}+\sqrt{2}\right)-\sqrt{3}\right\}}{5-\left(5-2\sqrt{6}\right)}$$

$$=\frac{\left(\sqrt{5}+\sqrt{2}\right)^2-3}{2\sqrt{6}}$$

$$=\frac{\left(7+2\sqrt{10}\right)-3}{2\sqrt{6}}$$

$$=\frac{4+2\sqrt{10}}{2\sqrt{6}}$$

$$=\frac{2+\sqrt{10}}{\sqrt{6}}$$

$$=\frac{\left(2+\sqrt{10}\right)}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}$$

$$=\frac{2\sqrt{6}+2\sqrt{15}}{6}$$

$$=\frac{\sqrt{6}+\sqrt{15}}{3}$$

(3)

$$\frac{\sqrt{2}+\sqrt{5}+\sqrt{7}}{\sqrt{2}+\sqrt{5}-\sqrt{7}}=\frac{\sqrt{2}+\sqrt{5}+\sqrt{7}}{\left(\sqrt{2}+\sqrt{5}\right)-\sqrt{7}}\cdot\frac{\left(\sqrt{2}+\sqrt{5}\right)+\sqrt{7}}{\left(\sqrt{2}+\sqrt{5}\right)+\sqrt{7}}$$

$$=\frac{\left(\sqrt{2}+\sqrt{5}+\sqrt{7}\right)^2}{\left(\sqrt{2}+\sqrt{5}\right)^2-\left(\sqrt{7}\right)^2}$$

$$=\frac{\left(\sqrt{2}+\sqrt{5}+\sqrt{7}\right)^2}{\left(7+2\sqrt{10}\right)-7}$$

$$=\frac{\left(\sqrt{2}+\sqrt{5}+\sqrt{7}\right)^2}{2\sqrt{10}}$$

$$=\frac{2+5+7+2\sqrt{10}+2\sqrt{35}+2\sqrt{14}}{2\sqrt{10}}$$

$$=\frac{7+\sqrt{10}+\sqrt{35}+\sqrt{14}}{\sqrt{10}}$$

$$\frac{\sqrt{2}-\sqrt{5}+\sqrt{7}}{\sqrt{2}-\sqrt{5}-\sqrt{7}}=\frac{\sqrt{2}-\sqrt{5}+\sqrt{7}}{\left(\sqrt{2}-\sqrt{5}\right)-\sqrt{7}}\cdot\frac{\left(\sqrt{2}-\sqrt{5}\right)+\sqrt{7}}{\left(\sqrt{2}-\sqrt{5}\right)+\sqrt{7}}$$
$$=\frac{\left(\sqrt{2}-\sqrt{5}+\sqrt{7}\right)^2}{\left(\sqrt{2}-\sqrt{5}\right)^2-\left(\sqrt{7}\right)^2}$$
$$=\frac{\left(\sqrt{2}-\sqrt{5}+\sqrt{7}\right)^2}{\left(7-2\sqrt{10}\right)-7}$$
$$=-\frac{\left(\sqrt{2}-\sqrt{5}+\sqrt{7}\right)^2}{2\sqrt{10}}$$
$$=-\frac{2+5+7-2\sqrt{10}-2\sqrt{35}+2\sqrt{14}}{2\sqrt{10}}$$
$$=-\frac{7-\sqrt{10}-\sqrt{35}+\sqrt{14}}{\sqrt{10}}$$
$$=\frac{-7+\sqrt{10}+\sqrt{35}-\sqrt{14}}{\sqrt{10}}$$

よって，

$$\frac{\sqrt{2}+\sqrt{5}+\sqrt{7}}{\sqrt{2}+\sqrt{5}-\sqrt{7}}+\frac{\sqrt{2}-\sqrt{5}+\sqrt{7}}{\sqrt{2}-\sqrt{5}-\sqrt{7}}=\frac{7+\sqrt{10}+\sqrt{35}+\sqrt{14}}{\sqrt{10}}+\frac{-7+\sqrt{10}+\sqrt{35}-\sqrt{14}}{\sqrt{10}}$$
$$=\frac{2\sqrt{10}+2\sqrt{35}}{\sqrt{10}}$$
$$=2+\frac{2\sqrt{7}}{\sqrt{2}}$$
$$=2+\sqrt{14}$$

61

$$\sqrt{x^2-10+25}=\sqrt{(x-5)^2}=|x-5|$$

(1)

$x-5\geq 0$ より，$|x-5|=x-5$

よって，$\sqrt{x^2-10+25}=x-5$

(2)

$x-5<0$ より，$|x-5|=-(x-5)=-x+5$

よって，$\sqrt{x^2-10+25}=-x+5$

62

(1)

$$\sqrt{4+2\sqrt{3}}=\sqrt{\left(\sqrt{3}\right)^2+1^2+2\cdot1\cdot\sqrt{3}}$$
$$=\sqrt{\left(1+\sqrt{3}\right)^2}$$
$$=\left|1+\sqrt{3}\right|$$
$$=1+\sqrt{3}$$

(2)

$$\sqrt{19-2\sqrt{48}}=\sqrt{\left(\sqrt{16}\right)^2+\left(\sqrt{3}\right)^2-2\cdot\sqrt{16}\cdot\sqrt{3}}$$
$$=\sqrt{\left(\sqrt{16}-\sqrt{3}\right)^2}$$
$$=\sqrt{\left(4-\sqrt{3}\right)^2}$$
$$=\left|4-\sqrt{3}\right|$$
$$=4-\sqrt{3}$$

(3)

$$\sqrt{9-2\sqrt{20}}=\sqrt{\left(\sqrt{5}\right)^2+\left(\sqrt{4}\right)^2+2\cdot\sqrt{5}\cdot\sqrt{4}}$$
$$=\sqrt{\left(\sqrt{4}+\sqrt{5}\right)^2}$$
$$=\sqrt{\left(2+\sqrt{5}\right)^2}$$
$$=\left|2+\sqrt{5}\right|$$
$$=2+\sqrt{5}$$

63

(1)

$$\sqrt{5+\sqrt{24}}=\sqrt{5+2\sqrt{6}}$$
$$=\sqrt{\left(\sqrt{2}\right)^2+\left(\sqrt{3}\right)^2+2\cdot\sqrt{2}\cdot\sqrt{3}}$$
$$=\sqrt{\left(\sqrt{2}+\sqrt{3}\right)^2}$$
$$=\left|\sqrt{2}+\sqrt{3}\right|$$
$$=\sqrt{2}+\sqrt{3}$$

(2)

$$\sqrt{11+4\sqrt{6}}=\sqrt{11+2\sqrt{24}}$$
$$=\sqrt{\left(\sqrt{3}\right)^2+\left(\sqrt{8}\right)^2+2\cdot\sqrt{3}\cdot\sqrt{8}}$$
$$=\sqrt{\left(\sqrt{3}+\sqrt{8}\right)^2}$$
$$=\sqrt{\left(\sqrt{3}+2\sqrt{2}\right)^2}$$
$$=\left|\sqrt{3}+2\sqrt{2}\right|$$
$$=\sqrt{3}+2\sqrt{2}$$

(3)

$$\sqrt{12-8\sqrt{2}}=\sqrt{12-2\sqrt{32}}$$
$$=\sqrt{\left(\sqrt{4}\right)^2+\left(\sqrt{8}\right)^2-2\cdot\sqrt{4}\cdot\sqrt{8}}$$
$$=\sqrt{\left(\sqrt{4}-\sqrt{8}\right)^2}$$
$$=\sqrt{\left(2-2\sqrt{2}\right)^2}$$
$$=\left|2-2\sqrt{2}\right|$$
$$=2\sqrt{2}-2$$

64

(1)

$$\sqrt{2+\sqrt{3}}=\sqrt{\frac{4+2\sqrt{3}}{2}}$$
$$=\frac{\sqrt{\left(\sqrt{1}\right)^2+\left(\sqrt{3}\right)^2+2\cdot1\cdot\sqrt{3}}}{\sqrt{2}}$$
$$=\frac{\sqrt{\left(\sqrt{1}+\sqrt{3}\right)^2}}{\sqrt{2}}$$
$$=\frac{\sqrt{\left(1+\sqrt{3}\right)^2}}{\sqrt{2}}$$
$$=\frac{\left|1+\sqrt{3}\right|}{\sqrt{2}}$$
$$=\frac{1+\sqrt{3}}{\sqrt{2}}$$
$$=\frac{\sqrt{2}+\sqrt{6}}{2}$$

(2)

$$\sqrt{5-\sqrt{21}}=\sqrt{\frac{10-2\sqrt{21}}{2}}$$
$$=\frac{\sqrt{\left(\sqrt{3}\right)^2+\left(\sqrt{7}\right)^2-2\cdot\sqrt{3}\cdot\sqrt{7}}}{\sqrt{2}}$$
$$=\frac{\sqrt{\left(\sqrt{3}-\sqrt{7}\right)^2}}{\sqrt{2}}$$
$$=\frac{\left|\sqrt{3}-\sqrt{7}\right|}{\sqrt{2}}$$
$$=\frac{\sqrt{7}-\sqrt{3}}{\sqrt{2}}$$
$$=\frac{\sqrt{14}-\sqrt{6}}{2}$$

(3)

$$\sqrt{10+5\sqrt{3}}=\sqrt{\frac{20+10\sqrt{3}}{2}}$$
$$=\sqrt{\frac{20+2\sqrt{75}}{2}}$$
$$=\frac{\sqrt{20+2\sqrt{75}}}{\sqrt{2}}$$
$$=\frac{\sqrt{\left(\sqrt{5}\right)^2+\left(\sqrt{15}\right)^2+2\cdot\sqrt{5}\cdot\sqrt{15}}}{\sqrt{2}}$$
$$=\frac{\sqrt{\left(\sqrt{5}+\sqrt{15}\right)^2}}{\sqrt{2}}$$
$$=\frac{\left|\sqrt{5}+\sqrt{15}\right|}{\sqrt{2}}$$
$$=\frac{\sqrt{5}+\sqrt{15}}{\sqrt{2}}$$
$$=\frac{\sqrt{10}+\sqrt{30}}{2}$$