式と証明 4　分数式とその計算

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(1)

$$\frac{2}{1+a}+\frac{4}{1+a^2}+\frac{2}{1-a}+\frac{8}{1+a^4}=\left(\frac{2}{1+a}+\frac{2}{1-a}\right)+\frac{4}{1+a^2}+\frac{8}{1+a^4}$$
$$=\frac{2(1-a)+2(1+a)}{(1+a)(1-a)}+\frac{4}{1+a^2}+\frac{8}{1+a^4}$$
$$=\frac{4}{1-a^2}+\frac{4}{1+a^2}+\frac{8}{1+a^4}$$
$$=\left(\frac{4}{1-a^2}+\frac{4}{1+a^2}\right)+\frac{8}{1+a^4}$$
$$=\frac{4\left(1+a^2\right)+4\left(1-a^2\right)}{\left(1-a^2\right)\left(1+a^2\right)}+\frac{8}{1+a^4}$$
$$=\frac{8}{1-a^4}+\frac{8}{1+a^4}$$
$$=\frac{8\left(1+a^4\right)+8\left(1-a^4\right)}{\left(1-a^4\right)\left(1+a^4\right)}$$
$$=\frac{16}{1-a^8}$$

(2)

$$\frac{ca}{(a-b)(b-c)}+\frac{ab}{(b-c)(c-a)}+\frac{bc}{(c-a)(a-b)}=\frac{ca(c-a)+ab(a-b)+bc(b-c)}{(a-b)(b-c)(c-a)}$$

ここで，分子をaについて整理すると，

$$ca(c-a)+ab(a-b)+bc(b-c)=c^2a-ca^2+a^2b-ab^2+bc(b-c)$$
$$=(b-c)a^2-\left(b^2-c^2\right)a+bc(b-c)$$
$$=(b-c)a^2-(b-c)(b+c)a+bc(b-c)$$
$$=(b-c)\left\{a^2-(b+c)a+bc\right\}$$
$$=(b-c)(a-b)(a-c)$$
$$=-(a-b)(b-c)(c-a)$$

よって，$\dfrac{ca}{(a-b)(b-c)}+\dfrac{ab}{(b-c)(c-a)}+\dfrac{bc}{(c-a)(a-b)}=-1$

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(1)

$$\frac{x+2}{x}-\frac{x+3}{x+1}-\frac{x-5}{x-3}+\frac{x-6}{x-4}=1+\frac{2}{x}-\frac{(x+1)+2}{x+1}-\frac{(x-3)-2}{x-3}+\frac{(x-4)-2}{x-4}$$

$$=1+\frac{2}{x}-\left(1+\frac{2}{x+1}\right)-\left(1-\frac{2}{x-3}\right)+\left(1-\frac{2}{x-4}\right)$$

$$=\frac{2}{x}-\frac{2}{x+1}+\frac{2}{x-3}-\frac{2}{x-4}$$

$$=\left(\frac{2}{x}-\frac{2}{x+1}\right)+\left(\frac{2}{x-3}-\frac{2}{x-4}\right)$$

$$=\frac{2(x+1)-2x}{x(x+1)}+\frac{2(x-4)-2(x-3)}{(x-3)(x-4)}$$

$$=\frac{2}{x(x+1)}-\frac{2}{(x-3)(x-4)}$$

$$=\frac{2(x-3)(x-4)-2x(x+1)}{x(x+1)(x-3)(x-4)}$$

$$=\frac{2(x^2-7x+12)-2(x^2+x)}{x(x+1)(x-3)(x-4)}$$

$$=\frac{-16x+24}{x(x+1)(x-3)(x-4)}$$

$$=-\frac{8(2x-3)}{x(x+1)(x-3)(x-4)}$$

(2)

$$\frac{2}{(a-1)(a+1)}=\frac{(a+1)-(a-1)}{(a-1)(a+1)}=\frac{1}{a-1}-\frac{1}{a+1}$$

$$\frac{2}{(a+1)(a+3)}=\frac{(a+3)-(a+1)}{(a+1)(a+3)}=\frac{1}{a+1}-\frac{1}{a+3}$$

$$\frac{2}{(a+3)(a+5)}=\frac{(a+5)-(a+3)}{(a+3)(a+5)}=\frac{1}{a+3}-\frac{1}{a+5}$$

より,

$$与式=\left(\frac{1}{a-1}-\frac{1}{a+1}\right)+\left(\frac{1}{a+1}-\frac{1}{a+3}\right)+\left(\frac{1}{a+3}-\frac{1}{a+5}\right)$$

$$=\frac{1}{a-1}-\frac{1}{a+5}$$

$$=\frac{a+5-(a-1)}{(a-1)(a+5)}$$

$$=\frac{6}{(a-1)(a+5)}$$

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$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$$
$$=4^2-2$$
$$=14$$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left\{x^2-x\cdot\frac{1}{x}+\left(\frac{1}{x}\right)^2\right\}$$
$$=4\left(x^2+\frac{1}{x^2}-1\right)$$
$$=4(13-1)$$
$$=52$$

または

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3x\cdot\frac{1}{x}\left(x+\frac{1}{x}\right)$$
$$=\left(x+\frac{1}{x}\right)\left\{\left(x+\frac{1}{x}\right)^2-3\right\}$$
$$=4\left(4^2-3\right)$$
$$=4\cdot 13$$
$$=52$$