三角関数 2　三角関数

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(1)

$$\text{左辺}=\frac{\sin^2\theta}{\tan^2\theta-\sin^2\theta}$$
$$=\frac{\sin^2\theta}{\dfrac{\sin^2\theta}{\cos^2\theta}-\sin^2\theta}$$
$$=\frac{1}{\dfrac{1}{\cos^2\theta}-1}$$
$$=\frac{1}{\left(1+\tan^2\theta\right)-1}$$
$$=\frac{1}{\tan^2\theta}$$
$$=\text{右辺}$$

(2)

$$\text{左辺}=\left(1+\sin\theta+\cos\theta\right)^2+\left(1+\sin\theta-\cos\theta\right)^2$$
$$=\left\{\left(1+\sin\theta\right)+\cos\theta\right\}+\left\{\left(1+\sin\theta\right)-\cos\theta\right\}^2$$
$$=2\left\{\left(1+\sin\theta\right)^2+\cos^2\theta\right\}$$
$$=2\left(1+2\sin\theta+\sin^2\theta+\cos^2\theta\right)$$
$$=2\left(1+2\sin\theta+1\right)$$
$$=4\left(1+\sin\theta\right)$$
$$=\text{右辺}$$

(3)

$$\text{左辺}=\frac{\cos^2\theta-\sin^2\theta}{1+2\sin\theta\cos\theta}$$
$$=\frac{\cos^2\theta\left(1-\dfrac{\sin^2\theta}{\cos^2\theta}\right)}{\cos^2\theta\left(\dfrac{1}{\cos^2\theta}+2\cdot\dfrac{\sin\theta}{\cos\theta}\right)}$$
$$=\frac{1-\tan^2\theta}{1+\tan^2\theta+2\tan\theta}$$
$$=\frac{\left(1+\tan\theta\right)\left(1-\tan\theta\right)}{\left(1+\tan\theta\right)^2}$$
$$=\frac{1-\tan\theta}{1+\tan\theta}$$
$$=\text{右辺}$$

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(1)

$$\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=\frac{2}{1-\sin^2\theta}$$
$$=\frac{2}{\cos^2\theta}$$
$$=2\cdot\frac{1}{\cos^2\theta}$$
$$=2\left(1+\tan^2\theta\right)$$
$$=2\left(1+2^2\right)$$
$$=10$$

(2)

$$\frac{1-\sin\theta}{\cos\theta}+\frac{\cos\theta}{1-\sin\theta}=\frac{(1-\sin\theta)^2+\cos^2\theta}{\cos\theta(1-\sin\theta)}$$
$$=\frac{1-2\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1-\sin\theta)}$$
$$=\frac{2(1-\sin\theta)}{\cos\theta(1-\sin\theta)}$$
$$=\frac{2}{\cos\theta}$$

ここで，$0<\theta<\frac{\pi}{2}$ より，$\cos\theta>0$

$$\therefore\frac{1-\sin\theta}{\cos\theta}+\frac{\cos\theta}{1-\sin\theta}=\frac{2}{\cos\theta}$$
$$=2\sqrt{\frac{1}{\cos^2\theta}}$$
$$=2\sqrt{1+\tan^2\theta}$$
$$=2\sqrt{1+5^2}$$
$$=2\sqrt{26}$$

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(1)

$$\tan\theta+\frac{1}{\tan\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$$
$$=\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}$$
$$=\frac{1}{\sin\theta\cos\theta}$$

ここで,

$$(\sin\theta+\cos\theta)^2=\frac{1}{4}$$

$$(\sin\theta+\cos\theta)^2=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$$
$$=1+2\sin\theta\cos\theta$$

より, $1+2\sin\theta\cos\theta=\frac{1}{4}\qquad\therefore\sin\theta\cos\theta=-\frac{3}{8}$

よって,

$$\tan\theta+\frac{1}{\tan\theta}=\frac{1}{\sin\theta\cos\theta}$$
$$=-\frac{8}{3}$$

(2)

$$\tan^3\theta+\frac{1}{\tan^3\theta}=\left(\tan\theta+\frac{1}{\tan\theta}\right)^3-3\tan\theta\cdot\frac{1}{\tan\theta}\left(\tan\theta+\frac{1}{\tan\theta}\right)$$
$$=\left(-\frac{8}{3}\right)^3-3\cdot\left(-\frac{8}{3}\right)$$
$$=-\frac{512}{27}+8$$
$$=-\frac{296}{27}$$

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(1)

$\pi < \theta < \frac{3}{2}\pi$ より， $\sin\theta < 0,\ \cos\theta < 0 \quad \therefore \sin\theta + \cos\theta < 0$ ・・・①

$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$$
$$= 1 + 2\sin\theta\cos\theta$$
$$= 1 + 2\cdot\frac{1}{4}$$
$$= \frac{3}{2}$$

よって，①より， $\sin\theta + \cos\theta = -\sqrt{\frac{3}{2}} = -\frac{\sqrt{6}}{2}$

(2)

$\sin\theta,\ \cos\theta$ を解とする t の 2 次方程式は $(t - \sin\theta)(t - \cos\theta) = 0$ より，

$t^2 - (\sin\theta + \cos\theta)t + \sin\theta\cos\theta = 0$

これと，(1)で， $1 + 2\sin\theta\cos\theta = \frac{3}{2}$ より， $\sin\theta\cos\theta = \frac{1}{4}$

また， $\sin\theta + \cos\theta = -\frac{\sqrt{6}}{2}$

よって， $t^2 + \frac{\sqrt{6}}{2}t + \frac{1}{4} = 0$，すなわち $4t^2 + 2\sqrt{6}t + 1 = 0 \quad \therefore t = \frac{-\sqrt{6} \pm \sqrt{2}}{4}$

ゆえに， $(\sin\theta,\ \cos\theta) = \left(-\frac{\sqrt{2}+\sqrt{6}}{4},\ \frac{\sqrt{2}-\sqrt{6}}{4}\right), \left(\frac{\sqrt{2}-\sqrt{6}}{4},\ -\frac{\sqrt{2}+\sqrt{6}}{4}\right)$

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解と係数の関係より， $\sin\theta + \cos\theta = \frac{7}{5}$ ・・・① $\sin\theta\cos\theta = \frac{k}{5}$ ・・・②

①より， $(\sin\theta + \cos\theta)^2 = \frac{49}{25}$

これと $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$ より，

$1 + 2\sin\theta\cos\theta = \frac{49}{25} \quad \therefore \sin\theta\cos\theta = \frac{12}{25}$

これを②に代入すると， $\frac{12}{25} = \frac{k}{5} \quad \therefore k = \frac{12}{5}$

これより，方程式は $5x^2 - 7x + \frac{12}{5} = 0 \Leftrightarrow \frac{1}{5}(25x^2 - 35x + 12) = 0 \Leftrightarrow \frac{1}{5}(5x-3)(5x-4) = 0$

$\therefore x = \frac{3}{5},\ \frac{4}{5}$

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$$\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}=\frac{\cos\theta\left(\frac{\sin\theta}{\cos\theta}+1\right)}{\cos\theta\left(\frac{\sin\theta}{\cos\theta}-1\right)}$$
$$=\frac{\tan\theta+1}{\tan\theta-1}$$

より，

$$\frac{\tan\theta+1}{\tan\theta-1}=3+2\sqrt{2}\quad \therefore\ \tan\theta+1=\left(3+2\sqrt{2}\right)\tan\theta-\left(3+2\sqrt{2}\right)$$

これを解くと $\tan\theta=\sqrt{2}$

よって，$1+\tan^2\theta=\frac{1}{\cos^2\theta}$ より，$\cos^2\theta=\frac{1}{1+\tan^2\theta}=\frac{1}{3}\quad \therefore \cos\theta=\pm\frac{\sqrt{3}}{3}$

さらに，$\sin^2\theta=1-\cos^2\theta$ より，$\sin^2=1-\frac{1}{3}=\frac{2}{3}\quad \therefore \sin\theta=\pm\frac{\sqrt{6}}{3}$

ここで，$\frac{\sin\theta}{\cos\theta}=\tan\theta=\sqrt{2}>0$ より，$\sin\theta$ と $\cos\theta$ は同符号である。

ゆえに，$(\sin\theta,\ \cos\theta,\ \tan\theta)=\left(\frac{\sqrt{6}}{3},\ \frac{\sqrt{3}}{3},\ \sqrt{2}\right),\left(-\frac{\sqrt{6}}{3},\ -\frac{\sqrt{3}}{3},\ \sqrt{2}\right)$

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$$1+\cos^2\theta+\cos^4\theta=1+\cos^2\theta+\left(\cos^2\theta\right)^2$$
$$=1+\left(1-\sin^2\theta\right)+\left(1-\sin^2\theta\right)^2$$
$$=1+\sin\theta+\sin^2\theta$$
$$=2$$