

三角関数 2 三角関数

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(1)

$$\begin{aligned}
 \text{左辺} &= \frac{\sin^2 \theta}{\tan^2 \theta - \sin^2 \theta} \\
 &= \frac{\sin^2 \theta}{\frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta} \\
 &= \frac{1}{\frac{1}{\cos^2 \theta} - 1} \\
 &= \frac{1}{(1 + \tan^2 \theta) - 1} \\
 &= \frac{1}{\tan^2 \theta} \\
 &= \text{右辺}
 \end{aligned}$$

(2)

$$\begin{aligned}
 \text{左辺} &= (1 + \sin \theta + \cos \theta)^2 + (1 + \sin \theta - \cos \theta)^2 \\
 &= \{(1 + \sin \theta) + \cos \theta\} + \{(1 + \sin \theta) - \cos \theta\}^2 \\
 &= 2\{(1 + \sin \theta)^2 + \cos^2 \theta\} \\
 &= 2(1 + 2\sin \theta + \sin^2 \theta + \cos^2 \theta) \\
 &= 2(1 + 2\sin \theta + 1) \\
 &= 4(1 + \sin \theta) \\
 &= \text{右辺}
 \end{aligned}$$

(3)

$$\begin{aligned}
 \text{左辺} &= \frac{\cos^2 \theta - \sin^2 \theta}{1 + 2\sin \theta \cos \theta} \\
 &= \frac{\cos^2 \theta \left(1 - \frac{\sin^2 \theta}{\cos^2 \theta}\right)}{\cos^2 \theta \left(\frac{1}{\cos^2 \theta} + 2 \cdot \frac{\sin \theta}{\cos \theta}\right)} \\
 &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta + 2 \tan \theta} \\
 &= \frac{(1 + \tan \theta)(1 - \tan \theta)}{(1 + \tan \theta)^2} \\
 &= \frac{1 - \tan \theta}{1 + \tan \theta} \\
 &= \text{右辺}
 \end{aligned}$$

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(1)

$$\begin{aligned}
 \frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} &= \frac{2}{1-\sin^2\theta} \\
 &= \frac{2}{\cos^2\theta} \\
 &= 2 \cdot \frac{1}{\cos^2\theta} \\
 &= 2(1+\tan^2\theta) \\
 &= 2(1+2^2) \\
 &= 10
 \end{aligned}$$

(2)

$$\begin{aligned}
 \frac{1-\sin\theta}{\cos\theta} + \frac{\cos\theta}{1-\sin\theta} &= \frac{(1-\sin\theta)^2 + \cos^2\theta}{\cos\theta(1-\sin\theta)} \\
 &= \frac{1-2\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1-\sin\theta)} \\
 &= \frac{2(1-\sin\theta)}{\cos\theta(1-\sin\theta)} \\
 &= \frac{2}{\cos\theta}
 \end{aligned}$$

ここで、 $0 < \theta < \frac{\pi}{2}$ より、 $\cos\theta > 0$

$$\begin{aligned}
 \therefore \frac{1-\sin\theta}{\cos\theta} + \frac{\cos\theta}{1-\sin\theta} &= \frac{2}{\cos\theta} \\
 &= 2\sqrt{\frac{1}{\cos^2\theta}} \\
 &= 2\sqrt{1+\tan^2\theta} \\
 &= 2\sqrt{1+5^2} \\
 &= 2\sqrt{26}
 \end{aligned}$$

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(1)

$$\begin{aligned}\tan \theta + \frac{1}{\tan \theta} &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{1}{\sin \theta \cos \theta}\end{aligned}$$

ここで,

$$(\sin \theta + \cos \theta)^2 = \frac{1}{4}$$

$$\begin{aligned}(\sin \theta + \cos \theta)^2 &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= 1 + 2 \sin \theta \cos \theta\end{aligned}$$

$$\text{より, } 1 + 2 \sin \theta \cos \theta = \frac{1}{4} \quad \therefore \sin \theta \cos \theta = -\frac{3}{8}$$

よって,

$$\begin{aligned}\tan \theta + \frac{1}{\tan \theta} &= \frac{1}{\sin \theta \cos \theta} \\ &= -\frac{8}{3}\end{aligned}$$

(2)

$$\begin{aligned}\tan^3 \theta + \frac{1}{\tan^3 \theta} &= \left(\tan \theta + \frac{1}{\tan \theta} \right)^3 - 3 \tan \theta \cdot \frac{1}{\tan \theta} \left(\tan \theta + \frac{1}{\tan \theta} \right) \\ &= \left(-\frac{8}{3} \right)^3 - 3 \cdot \left(-\frac{8}{3} \right) \\ &= -\frac{512}{27} + 8 \\ &= -\frac{296}{27}\end{aligned}$$

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(1)

$$\pi < \theta < \frac{3}{2}\pi \text{ より, } \sin \theta < 0, \cos \theta < 0 \quad \therefore \sin \theta + \cos \theta < 0 \quad \dots \textcircled{1}$$

$$\begin{aligned} (\sin \theta + \cos \theta)^2 &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= 1 + 2 \sin \theta \cos \theta \\ &= 1 + 2 \cdot \frac{1}{4} \\ &= \frac{3}{2} \end{aligned}$$

$$\text{よって, } \textcircled{1} \text{ より, } \sin \theta + \cos \theta = -\sqrt{\frac{3}{2}} = -\frac{\sqrt{6}}{2}$$

(2)

$\sin \theta, \cos \theta$ を解とする t の 2 次方程式は $(t - \sin \theta)(t - \cos \theta) = 0$ より,

$$t^2 - (\sin \theta + \cos \theta)t + \sin \theta \cos \theta = 0$$

$$\text{これと, (1) で, } 1 + 2 \sin \theta \cos \theta = \frac{3}{2} \text{ より, } \sin \theta \cos \theta = \frac{1}{4}$$

$$\text{また, } \sin \theta + \cos \theta = -\frac{\sqrt{6}}{2}$$

$$\text{よって, } t^2 + \frac{\sqrt{6}}{2}t + \frac{1}{4} = 0, \text{ すなわち } 4t^2 + 2\sqrt{6}t + 1 = 0 \quad \therefore t = \frac{-\sqrt{6} \pm \sqrt{2}}{4}$$

$$\text{ゆえに, } (\sin \theta, \cos \theta) = \left(-\frac{\sqrt{2} + \sqrt{6}}{4}, \frac{\sqrt{2} - \sqrt{6}}{4} \right), \left(\frac{\sqrt{2} - \sqrt{6}}{4}, -\frac{\sqrt{2} + \sqrt{6}}{4} \right)$$

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$$\text{解と係数の関係より, } \sin \theta + \cos \theta = \frac{7}{5} \quad \dots \textcircled{1} \quad \sin \theta \cos \theta = \frac{k}{5} \quad \dots \textcircled{2}$$

$$\textcircled{1} \text{ より, } (\sin \theta + \cos \theta)^2 = \frac{49}{25}$$

$$\text{これと } (\sin \theta + \cos \theta)^2 = \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta = 1 + 2 \sin \theta \cos \theta \text{ より,}$$

$$1 + 2 \sin \theta \cos \theta = \frac{49}{25} \quad \therefore \sin \theta \cos \theta = \frac{12}{25}$$

$$\text{これを} \textcircled{2} \text{ に代入すると, } \frac{12}{25} = \frac{k}{5} \quad \therefore k = \frac{12}{5}$$

$$\text{これより, 方程式は } 5x^2 - 7x + \frac{12}{5} = 0 \Leftrightarrow \frac{1}{5}(25x^2 - 35x + 12) = 0 \Leftrightarrow \frac{1}{5}(5x - 3)(5x - 4) = 0$$

$$\therefore x = \frac{3}{5}, \frac{4}{5}$$

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$$\frac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta} = \frac{\cos \theta \left(\frac{\sin \theta}{\cos \theta} + 1 \right)}{\cos \theta \left(\frac{\sin \theta}{\cos \theta} - 1 \right)}$$

$$= \frac{\tan \theta + 1}{\tan \theta - 1}$$

より,

$$\frac{\tan \theta + 1}{\tan \theta - 1} = 3 + 2\sqrt{2} \quad \therefore \tan \theta + 1 = (3 + 2\sqrt{2})\tan \theta - (3 + 2\sqrt{2})$$

これを解くと $\tan \theta = \sqrt{2}$

$$\text{よって, } 1 + \tan^2 \theta = \frac{1}{\cos^2 \theta} \text{ より, } \cos^2 \theta = \frac{1}{1 + \tan^2 \theta} = \frac{1}{3} \quad \therefore \cos \theta = \pm \frac{\sqrt{3}}{3}$$

$$\text{さらに, } \sin^2 \theta = 1 - \cos^2 \theta \text{ より, } \sin^2 \theta = 1 - \frac{1}{3} = \frac{2}{3} \quad \therefore \sin \theta = \pm \frac{\sqrt{6}}{3}$$

ここで, $\frac{\sin \theta}{\cos \theta} = \tan \theta = \sqrt{2} > 0$ より, $\sin \theta$ と $\cos \theta$ は同符号である。

$$\text{ゆえに, } (\sin \theta, \cos \theta, \tan \theta) = \left(\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3}, \sqrt{2} \right), \left(-\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}, \sqrt{2} \right)$$

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$$1 + \cos^2 \theta + \cos^4 \theta = 1 + \cos^2 \theta + (\cos^2 \theta)^2$$

$$= 1 + (1 - \sin^2 \theta) + (1 - \sin^2 \theta)^2$$

$$= 1 + \sin \theta + \sin^2 \theta$$

$$= 2$$