積分法 4　いろいろな関数の不定積分

超おすすめ問題集

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(1)

$$\frac{x^2+x+1}{x^2+1}=\frac{(x^2+1)+x}{x^2+1}$$
$$=1+\frac{x}{x^2+1}$$
$$=1+\frac{\frac{1}{2}(x^2+1)'}{x^2+1}$$
$$=1+\frac{1}{2}\cdot\frac{(x^2+1)'}{x^2+1}$$

より，

$$\int\frac{x^2+x+1}{x^2+1}dx=\int\left\{1+\frac{1}{2}\cdot\frac{(x^2+1)'}{x^2+1}\right\}dx$$
$$=x+\frac{1}{2}\log(x^2+1)+C$$

(2)

$$\frac{x^4}{x^2-1}=\frac{(x^2+1)(x^2-1)+1}{x^2-1}$$
$$=x^2+1+\frac{1}{x^2-1}$$
$$=x^2+1+\frac{1}{(x-1)(x+1)}$$
$$=x^2+1+\frac{1}{2}\cdot\frac{(x+1)-(x-1)}{(x-1)(x+1)}$$
$$=x^2+1+\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$$

より，

$$\int\frac{x^4}{x^2-1}dx=\int\left\{x^2+1+\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\right\}dx$$
$$=\frac{1}{3}x^3+x+\frac{1}{2}\left(\log|x-1|+\log|x+1|\right)+C$$
$$=\frac{1}{3}x^3+x+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$$

(3)

$$\frac{x^3}{x^2-4}=\frac{x(x^2-4)+4x}{x^2-4}$$
$$=x+\frac{4x}{x^2-4}$$
$$=x+2\cdot\frac{(x^2-4)'}{x^2-4}$$

より，

$$\int\frac{x^3}{x^2-4}dx=\int\left\{x+2\cdot\frac{(x^2-4)'}{x^2-4}\right\}dx$$
$$=\frac{1}{2}x^2+2\log\left|x^2-4\right|+C$$

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(1)

両辺を $x(x+1)^2$ 倍すると，

$3x+2=a(x+1)^2+bx(x+1)+cx$

$x=-1$ を代入すると， $-1=-c$ 　$\therefore c=1$ 　・・・①

$x=0$ を代入すると， $2=a$ 　$\therefore a=2$ 　・・・②

$x=1$ を代入すると， $5=4a+2b+c$

これと①，②より， $5=8+2b+1$ 　$\therefore b=-2$

逆に， $a=2,\ b=-2,\ c=1$ とすると，等式が成り立つ。

よって， $a=2,\ b=-2,\ c=1$

(2)

$$\int\frac{3x+2}{x(x+1)^2}dx=\int\left\{\frac{2}{x}+\frac{-2}{x+1}+\frac{1}{(x+1)^2}\right\}dx$$
$$=2\log|x|-2\log|x+1|+\int(x+1)^{-2}dx$$
$$=2\log\left|\frac{x}{x+1}\right|-\frac{1}{x+1}+C$$

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(1)

$\dfrac{1}{x(x^2-1)}=\dfrac{1}{x(x-1)(x+1)}=\dfrac{a}{x}+\dfrac{b}{x-1}+\dfrac{c}{x+1}$ とすると,

$1=a(x-1)(x+1)+bx(x+1)+cx(x-1)$

$x=-1,0,1$ を代入することにより, $a=-1,\ b=\dfrac{1}{2},\ c=\dfrac{1}{2}$ が得られるから,

$$\int\frac{dx}{x(x^2-1)}=\int\left(-\frac{1}{x}+\frac{1}{2}\cdot\frac{1}{x-1}+\frac{1}{2}\cdot\frac{1}{x+1}\right)dx$$

$$=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C$$

$$=-\frac{1}{2}\log x^2+\frac{1}{2}\log|x^2-1|+C$$

$$=\frac{1}{2}\log\frac{|x^2-1|}{x^2}+C$$

(2)

$\dfrac{1}{x^2(x+2)}=\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+2}$ とおくと, $1=ax(x+2)+b(x+2)+cx^2$

$x=-2,\ -1,\ 0$ を代入することにより, $a=-\dfrac{1}{4},\ b=\dfrac{1}{2},\ c=\dfrac{1}{4}$

よって,

$$\int\frac{1}{x^2(x+2)}dx=\int\left(-\frac{1}{4}\cdot\frac{1}{x}+\frac{1}{2}x^{-2}+\frac{1}{4}\cdot\frac{1}{x+2}\right)dx$$

$$=-\frac{1}{4}\log|x|-\frac{1}{2x}+\frac{1}{4}\log|x+2|+C$$

$$=\frac{1}{4}\log\left|\frac{x+2}{x}\right|-\frac{1}{2x}+C$$

(3)

$\dfrac{1}{x(x^2+1)}=\dfrac{a}{x}+\dfrac{bx+c}{x^2+1}$ とおくと， $1=a(x^2+1)+x(bx+c)=(a+b)x^2+cx+a$

よって， $a+b=0,\ c=0,\ a=1$ より， $a=1,\ b=-1,\ c=0$

ゆえに，

$$\begin{aligned}\int\frac{1}{x(x^2+1)}dx&=\int\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx\\&=\int\left\{\frac{1}{x}-\frac{1}{2}\cdot\frac{(x^2+1)'}{x^2+1}\right\}dx\\&=\log|x|-\frac{1}{2}\log(x^2+1)+C\\&=\frac{1}{2}\log x^2-\frac{1}{2}\log(x^2+1)+C\\&=\frac{1}{2}\log\frac{x^2}{x^2+1}+C\end{aligned}$$

(4)

$\dfrac{x^2+1}{x^4-5x^2+4}=\dfrac{x^2+1}{(x^2-1)(x^2-4)}=\dfrac{a}{x^2-1}+\dfrac{b}{x^2-4}$ とおくと，

$x^2+1=a(x^2-4)+b(x^2-1)=(a+b)x^2-(4a+b)$

よって， $a+b=1,\ 4a+b=-1$ より， $a=-\dfrac{2}{3},\ b=\dfrac{5}{3}$

また，

$$\frac{1}{x^2-1}=\frac{1}{(x-1)(x+1)}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$$

$$\frac{1}{x^2-4}=\frac{1}{(x-2)(x+2)}=\frac{1}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)$$

ゆえに，

$$\begin{aligned}\frac{x^2+1}{x^4-5x^2+4}&=-\frac{2}{3}\cdot\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)+\frac{5}{3}\cdot\frac{1}{4}\left(\frac{1}{x-2}+\frac{1}{x+2}\right)\\&=-\frac{1}{3}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)+\frac{5}{12}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)\end{aligned}$$

$$\begin{aligned}\therefore\int\frac{x^2+1}{x^4-5x^2+4}dx&=-\frac{1}{3}\left(\log|x-1|-\log|x+1|\right)+\frac{5}{12}\left(\log|x-2|-\log|x+2|\right)+C\\&=-\frac{1}{3}\log\left|\frac{x-1}{x+1}\right|+\frac{5}{12}\log\left|\frac{x-2}{x+2}\right|+C\end{aligned}$$

(5)

$\dfrac{3x+2}{x(x+1)^3}=\dfrac{a}{x}+\dfrac{b}{x+1}+\dfrac{c}{(x+1)^2}+\dfrac{d}{(x+1)^3}$ とおくと，

$3x+2=a(x+1)^3+bx(x+1)^2+cx(x+1)+dx$

$x=0$ を代入すると， $2=a$ $\quad\therefore a=2$

$x=-1$ を代入すると， $-1=-d$ $\quad\therefore d=1$

$x=1$ を代入すると， $5=8a+4b+2c+d=16+4b+2c+1$ $\quad\therefore 2b+c=-6$

$x=-2$ を代入すると， $-4=-a-2b+2c-2d=-2b+2c-4$ $\quad\therefore b=c$

よって， $a=2,\ b=c=-2,\ d=1$

$$\int\frac{3x+2}{x(x+1)^3}dx=\int\left\{\frac{2}{x}-\frac{2}{x+1}-2(x+1)^{-2}+(x+1)^{-3}\right\}dx$$

$$=2\log|x|-2\log|x+1|+2(x+1)^{-1}-\frac{1}{2}(x+1)^{-2}+C$$

$$=\log\frac{x^2}{(x+1)^2}+\frac{2}{x+1}-\frac{1}{2(x+1)^2}+C$$

$$=\log\frac{x^2}{(x+1)^2}+\frac{4x+3}{2(x+1)^2}+C$$

(6)

$$\frac{x^4}{x^3-3x+2}=\frac{x(x^3-3x+2)+3x^2-2x}{x^3-3x+2}$$

$$=x+\frac{3x^2-2x}{x^3-3x+2}$$

$$=x+\frac{3x^2-2x}{(x-1)^2(x+2)}$$

ここで， $\dfrac{3x^2-2x}{(x-1)^2(x+2)}=\dfrac{a}{x+2}+\dfrac{b}{x-1}+\dfrac{c}{(x-1)^2}$ とおくと，

$3x^2-2x=a(x-1)^2+b(x-1)(x+2)+c(x+2)$

$x=-2$ を代入すると， $16=9a$ $\quad\therefore a=\dfrac{16}{9}$

$x=1$ を代入すると， $1=3c$ $\quad\therefore c=\dfrac{1}{3}$

$x=0$ を代入すると， $0=a-2b+2c=\dfrac{16}{9}-2b+\dfrac{2}{3}=\dfrac{22}{9}-2b$ $\quad\therefore b=\dfrac{11}{9}$

よって， $\dfrac{x^4}{x^3-3x+2}=x+\dfrac{16}{9}\cdot\dfrac{1}{x+2}+\dfrac{11}{9}\cdot\dfrac{1}{x-1}+\dfrac{1}{3}(x-1)^{-2}$

ゆえに，

$$\int \frac{x^4}{x^3-3x+2}dx = \int \left\{ x + \frac{16}{9} \cdot \frac{1}{x+2} + \frac{11}{9} \cdot \frac{1}{x-1} + \frac{1}{3}(x-1)^{-2} \right\} dx$$

$$= \frac{1}{2}x^2 + \frac{16}{9}\log|x+2| + \frac{11}{9}\log|x-1| - \frac{1}{3}(x+1)^{-1} + C$$

$$= \frac{1}{2}x^2 + \frac{16}{9}\log|x+2| + \frac{11}{9}\log|x-1| - \frac{1}{3(x-1)} + C$$

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(1)

$$\frac{1}{\sqrt{x+1}-\sqrt{x}} = \frac{\sqrt{x+1}+\sqrt{x}}{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)} = \sqrt{x+1}+\sqrt{x} = (x+1)^{\frac{1}{2}} + x^{\frac{1}{2}}$$

$$\therefore \int \frac{dx}{\sqrt{x+1}-\sqrt{x}} = \int \left\{ (x+1)^{\frac{1}{2}} + x^{\frac{1}{2}} \right\} dx$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$$

$$= \frac{2}{3}\left\{ (x+1)\sqrt{x+1} + x\sqrt{x} \right\} + C$$

(2)

$$\frac{x}{\sqrt{3x+4}-2} = \frac{x\left(\sqrt{3x+4}+2\right)}{\left(\sqrt{3x+4}-2\right)\left(\sqrt{3x+4}+2\right)} = \frac{1}{3}\left\{ (3x+4)^{\frac{1}{2}} + 2 \right\}$$

$$\therefore \int \frac{x}{\sqrt{3x+4}-2}dx = \frac{1}{3}\int \left\{ (3x+4)^{\frac{1}{2}} + 2 \right\} dx$$

$$= \frac{1}{3}\left\{ \frac{2}{9}(3x+1)^{\frac{3}{2}} + 2x \right\} + C$$

$$= \frac{2}{27}(3x+4)\sqrt{3x+4} + \frac{2}{3}x + C$$

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(1)

$$\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+1}dx$$

$x^{\frac{1}{4}} = t$ とおくと， $x = t^4$ より， $dx = 4t^3 dt$

$$\begin{aligned}
\therefore \int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx &= \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+1}dx \\
&= \int \frac{t^2}{t^3+1}\cdot 4t^3 dt \\
&= \int \frac{4t^5}{t^3+1}dt \\
&= \int \frac{4t^2(t^3+1)-4t^2}{t^3+1}dt \\
&= \int \left(4t^2 - \frac{4t^2}{t^3+1}\right)dt \\
&= \int \left\{4t^2 - \frac{4}{3}\cdot\frac{(t^3+1)'}{t^3+1}\right\}dt \\
&= \frac{4}{3}t^3 - \frac{4}{3}\log|t^3+1| + C \\
&= \frac{4}{3}x^{\frac{3}{4}} - \frac{4}{3}\log\left|x^{\frac{3}{4}}+1\right| + C \\
&= \frac{4}{3}\sqrt[4]{x^3} - \frac{4}{3}\log\left|\sqrt[4]{x^3}+1\right| + C \\
&= \frac{4}{3}\sqrt[4]{x^3} - \frac{4}{3}\log\left(\sqrt[4]{x^3}+1\right) + C \quad (\because x > 0)
\end{aligned}$$

(2)

$\sqrt{x+1}=t$ とおくと，$x+1=t^2$ より，$dx=2tdt$，$x=t^2-1$

$$\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2tdt}{(t^2-1)t}$$
$$=\int\frac{2}{t^2-1}dt$$
$$=\int\frac{2}{(t-1)(t+1)}dt$$
$$=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$
$$=\log|t-1|-\log|t+1|+C$$
$$=\log\left|\frac{t-1}{t+1}\right|+C$$
$$=\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$$

(3)

$$\int\log|x^2-1|dx=\int(\log|x-1|+\log|x+1|)dx$$
$$=\int\log|x-1|dx+\int\log|x+1|dx$$
$$=\int(x-1)'\log|x-1|dx+\int(x+1)'\log|x+1|dx$$
$$=(x-1)\log|x-1|-\int(x-1)\{\log|x-1|\}'dx+(x+1)\log|x+1|-\int(x+1)\{\log|x+1|\}'dx$$
$$=(x-1)\log|x-1|-\int dx+(x+1)\log|x+1|-\int dx$$
$$=(x-1)\log|x-1|+(x+1)\log|x+1|-2x+C$$

(4)

$$\int\frac{e^x}{e^x-e^{-x}}dx=\int\frac{e^x\cdot e^x}{(e^x-e^{-x})e^x}dx$$
$$=\int\frac{e^{2x}}{e^{2x}-1}dx$$
$$=\int\frac{e^{2x}}{(e^x-1)(e^x+1)}dx$$
$$=\frac{1}{2}\int\left(\frac{e^x}{e^x-1}+\frac{e^x}{e^x+1}\right)dx$$
$$=\frac{1}{2}\{\log|e^x-1|+\log|e^x+1|\}+C$$
$$=\frac{1}{2}\log|e^{2x}-1|+C$$

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(1)

$\sin x+1=t$ とおくと， $\cos xdx=dt$

$$\int\frac{\cos x}{\sin x(\sin x+1)}dx=\int\frac{dt}{(t-1)t}$$
$$=\int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$$
$$=\log|t-1|-\log|t|+C$$
$$=\log\left|\frac{t-1}{t}\right|+C$$
$$=\log\left|\frac{\sin x}{\sin x+1}\right|+C$$
$$=\log\frac{|\sin x|}{\sin x+1}+C$$

(2)

ポイント： $\int\cos x\sin^n xdx=\frac{1}{n+1}\sin^{n+1}x+C$， $\int\sin x\cos^n xdx=-\frac{1}{n+1}\cos^{n+1}x+C$

解説

$\int\cos x\sin^n xdx=\frac{1}{n+1}\sin^{n+1}x+C$ について

$(\sin^{n+1}x)'=(n+1)\cos x\sin^n x$ より， $\cos x\sin^n x=\frac{1}{n+1}(\sin^{n+1}x)'$

よって， $\int\cos x\sin^n xdx=\frac{1}{n+1}\sin^{n+1}x+C$

$\int\sin x\cos^n xdx=-\frac{1}{n+1}\cos^{n+1}x+C$ について

$(\cos^{n+1}x)'=-(n+1)\sin x\cos^n x$ より， $\sin x\cos^n x=-\frac{1}{n+1}(\cos^{n+1}x)'$

よって， $\int\sin x\cos^n xdx=-\frac{1}{n+1}\cos^{n+1}x+C$

解

$$\int\sin^2 x\cos^3 xdx=\int\sin^2 x\cos^2 x\cos xdx$$
$$=\int\sin^2 x(1-\sin^2 x)\cos xdx$$
$$=\int(\sin^2 x\cos x-\sin^4 x\cos x)dx$$
$$=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$$

$\int\cos x\sin^n x=\frac{1}{n+1}\sin^{n+1}x+C$ が利用できるように式変形

(3)

解法 **1**：置換積分を利用

$\tan x = t$ とおくと，$\dfrac{dx}{\cos^2 x} = dt$ より，$dx = \cos^2 xdt$ ・・・①

$1 + \tan^2 x = \dfrac{1}{\cos^2 x}$ だから，$\cos^2 x = \dfrac{1}{1+\tan^2 x} = \dfrac{1}{1+t^2}$ ・・・②

①，②より，$dx = \dfrac{1}{1+t^2}dt$

$$\begin{aligned}
\therefore \int \tan^4 xdx &= \int \frac{t^4}{1+t^2}dt \\
&= \int \frac{(t^2-1)(t^2+1)+1}{1+t^2}dt \\
&= \int \left(t^2 - 1 + \frac{1}{1+t^2}\right)dt \\
&= \int (t^2-1)dt + \int \frac{1}{1+t^2}dt \\
&= \frac{1}{3}t^3 - t + \int \frac{1}{1+\tan^2 x}\cdot\frac{dx}{\cos^2 x} \\
&= \frac{1}{3}\tan^3 x - \tan x + \int \frac{1}{\frac{1}{\cos^2 x}}\frac{dx}{\cos^2 x} \\
&= \frac{1}{3}\tan^3 x - \tan x + \int dx \\
&= \frac{1}{3}\tan^3 x - \tan x + x + C
\end{aligned}$$

解法 **2**：ヒントを利用

$$\begin{aligned}
\int \tan^4 xdx &= \int \tan^2 x \tan^2 xdx \\
&= \int \tan^2 x\left(\frac{1}{\cos^2 x} - 1\right)dx \\
&= \int \left(\frac{\tan^2 x}{\cos^2 x} - \tan^2 x\right)dx \\
&= \int \left\{\tan^2 x \cdot \frac{1}{\cos^2 x} - \left(\frac{1}{\cos^2 x} - 1\right)\right\}dx \\
&= \int \left\{\tan^2 x(\tan x)' - (\tan x)' + 1\right\}dx \\
&= \int \left\{\left(\frac{1}{3}\tan^3 x\right)' - (\tan x)' + 1\right\}dx \\
&= \frac{1}{3}\tan^3 x - \tan x + x + C
\end{aligned}$$

(4)

$1+\sin x=t$ とおくと，　$\cos x dx=dt$

よって，

$$\int\frac{\sin 2x}{1+\sin x}dx=\int\frac{2\sin x\cos x}{1+\sin x}dx$$
$$=\int\frac{2\sin x}{1+\sin x}\cdot\cos x dx$$
$$=\int\frac{2(t-1)}{t}dt$$
$$=2\int\left(1-\frac{1}{t}\right)dt$$
$$=2(t-\log|t|)+C'$$
$$=2(\sin x+1-\log|1+\sin x|)+C'$$
$$=2\sin x-2\log|1+\sin x|+C$$

(5)

$$\int\frac{1}{1-\sin x}dx=\int\frac{1+\sin x}{(1-\sin x)(1+\sin x)}dx$$
$$=\int\frac{1+\sin x}{\cos^2 x}dx$$
$$=\int\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx$$
$$=\int\left\{(\tan x)'+\left(\frac{1}{\cos x}\right)'\right\}dx$$
$$=\tan x+\frac{1}{\cos x}+C$$

(6)

解法 **1**：因数分解してから整理

$$\sin^3 x-\cos^3 x=(\sin x-\cos x)(\sin^2 x-\sin x\cos x+\cos^2 x)$$
$$=(\sin x-\cos x)(1+\sin x\cos x)$$
$$=\sin x-\cos x+\sin^2 x\cos x-\sin x\cos^2 x$$
$$=-(\cos x)'-(\sin x)'+\frac{1}{3}(\sin^3 x)'+\frac{1}{3}(\cos^3 x)'$$

よって，

$$\int(\sin^3 x-\cos^3 x)dx=-\cos x-\sin x+\frac{1}{3}\sin^3 x+\frac{1}{3}\cos^3 x+C$$

解法 **2**：**3** 倍角の公式を利用

$\sin 3x = 3\sin x - 4\sin^3 x \quad \therefore \sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

$\cos 3x = -3\cos x + 4\cos^3 x \quad \therefore \cos^3 x = \dfrac{3\cos x + \cos 3x}{4}$

より，

$\sin^3 x - \cos^3 x = \dfrac{3}{4}(\sin x - \cos x) - \dfrac{1}{4}(\sin 3x + \cos 3x)$

よって，

$$\begin{aligned}\int(\sin^3 x - \cos^3 x)dx &= \frac{3}{4}(-\cos x - \sin x) - \frac{1}{12}(-\cos 3x + \sin 3x) + C \\ &= -\frac{3}{4}(\sin x + \cos x) - \frac{1}{12}(\sin 3x - \cos 3x) + C\end{aligned}$$

補足：**3** 倍角の公式の覚え方の工夫例

表にして覚える

$\sin 3x$	$3\sin x$	$-4\sin^3 x$
$\cos 3x$	$-3\cos x$	$4\cos^3 x$

三角関数の積分を有理関数の積分に変換する一般的方法

$\tan\dfrac{x}{2}=t$ とおくと，

$$\tan x=\frac{2t}{1-t^2}\,,\quad \sin x=\frac{2t}{1+t^2}\,,\quad \cos x=\frac{1-t^2}{1+t^2}\,,\qquad dx=\frac{2}{1+t^2}dt$$

注意

却って計算が煩雑になることもある。おそらく入試ではそういう問題が出るでしょう。

解説

$$\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{2t}{1-t^2}$$

$$\sin x=\frac{\sin x}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\dfrac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=\frac{2\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1+\left(\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos x}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\dfrac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\dfrac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=\frac{1-\left(\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2}{1+\left(\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

$$\frac{d}{dx}\tan\frac{x}{2}=\frac{dt}{dx}\qquad\cdot\cdot\cdot①$$

$$\frac{d}{dx}\tan\frac{x}{2}=\frac{1}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}=\frac{1}{2}\cdot\frac{1}{\dfrac{1}{1+\tan^2\frac{x}{2}}}=\frac{1+\tan^2\frac{x}{2}}{2}=\frac{1+t^2}{2}\qquad\cdot\cdot\cdot②$$

①，②より，$\dfrac{1+t^2}{2}=\dfrac{dt}{dx}$

よって，$dx=\dfrac{2}{1+t^2}dt$

例 1

$$\int \frac{1}{1-\sin x}dx = \int \frac{1}{1-\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}dt$$
$$= \int \frac{2}{(1-t)^2}dt$$
$$= \frac{2}{1-t} + C$$
$$= \frac{2}{1-\tan\frac{x}{2}} + C$$
$$= -\frac{2}{\tan\frac{x}{2}-1} + C$$

例 2

$$\int \frac{1}{\cos x}dx = \int \frac{1}{\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt$$
$$= \int \frac{2}{1-t^2}dt$$
$$= \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt$$
$$= -\log|1-t| + \log|1+t| + C$$
$$= \log\left|\frac{1+t}{1-t}\right| + C$$
$$= \log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$$

例 3

$$\int \frac{1}{1+\sin x+\cos x}dx = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}dt$$
$$= \int \frac{1}{1+t}dt$$
$$= \log|1+t| + C$$
$$= \log\left|1+\tan\frac{x}{2}\right| + C$$

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(1)

解法 **1**：例題の解き方

$$\int e^x \cos x dx = \int e^x (\sin x)' dx$$
$$= e^x \sin x - \int (e^x)' \sin x dx$$
$$= e^x \sin x - \int e^x \sin x dx$$
$$= e^x \sin x + \int e^x (\cos x)' dx$$
$$= e^x \sin x + e^x \cos x - \int (e^x)' \cos x dx$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

よって，$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C$

解法 **2**：有名な解き方

$(e^x \sin x)' = e^x \sin x + e^x \cos x$　・・・①

$(e^x \cos x)' = -e^x \sin x + e^x \cos x$　・・・②

①＋②より，$(e^x \sin x)' + (e^x \cos x)' = 2e^x \cos x$　$\therefore e^x \cos x = \left\{\frac{1}{2} e^x (\sin x + \cos x)\right\}'$

ゆえに，$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C$

(2)

解法 **1**：例題の解き方

$$\int e^{-x} \sin x dx = -\int e^{-x} (\cos x)' dx$$
$$= -\left\{ e^{-x} \cos x - \int (e^{-x})' \cos x dx \right\}$$
$$= -e^{-x} \cos x - \int e^{-x} \cos x dx$$
$$= -e^{-x} \cos x - \int e^{-x} (\sin x)' dx$$
$$= -e^{-x} \cos x - \left\{ e^{-x} \sin x - \int (e^{-x})' \sin x dx \right\}$$
$$= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x dx$$

よって，$\int e^{-x} \sin x dx = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C$

解法 2：有名な解き方

$\left(e^{-x}\sin x\right)' = -e^{-x}\sin x + e^{-x}\cos x$ ・・・①

$\left(e^{-x}\cos x\right)' = -e^{-x}\sin x - e^{-x}\cos x$ ・・・②

①＋②より，$\left(e^{-x}\sin x\right)' + \left(e^{-x}\cos x\right)' = -2e^{-x}\sin x$ $\therefore e^{-x}\sin x = \left\{-\frac{1}{2}e^{x}(\sin x + \cos x)\right\}'$

ゆえに，$\int e^{-x}\sin x dx = -\frac{1}{2}e^{x}(\sin x + \cos x) + C$

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(1)

$\cos x = t$ とおくと，$-\sin x = \frac{dt}{dx}$ より，$\sin x dx = -dt$

よって，

$$\begin{aligned}\int \sin x \log(\cos x)dx &= -\int \log t dt \\ &= -\int t' \log t dt \\ &= -\left\{t\log t - \int t(\log t)' dt\right\} \\ &= -\left(t\log t - \int dt\right) \\ &= -t\log t + t + C \\ &= -\cos x\log(\cos x) + \cos x + C\end{aligned}$$

(2)

$$\begin{aligned}\int x\tan^2 x dx &= \int x\left(\frac{1}{\cos^2 x} - 1\right)dx \\ &= \int x\left\{(\tan x)' - 1\right\}dx \\ &= \int x(\tan x)' dx - \int x dx \\ &= x\tan x - \int x'\tan x dx - \frac{1}{2}x^2 \\ &= x\tan x - \int \tan x dx - \frac{1}{2}x^2 \\ &= x\tan x - \int \frac{\sin x}{\cos x}dx - \frac{1}{2}x^2 \\ &= x\tan x + \int \frac{(\cos x)'}{\cos x} - \frac{1}{2}x^2 \\ &= x\tan x + \log|\cos x| - \frac{1}{2}x^2 + C\end{aligned}$$

(3)

解法 1

$$\int x^2 \log(x+1)dx = \int \left(\frac{x^3+1}{3}\right)' \log(x+1)dx$$

$$= \frac{1}{3}\int (x^3+1)' \log(x+1)dx$$

$$= \frac{1}{3}\left[(x^3+1)\log(x+1) - \int (x^3+1)\{\log(x+1)\}' dx\right]$$

$$= \frac{1}{3}\left\{(x^3+1)\log(x+1) - \int \frac{x^3+1}{x+1}dx\right\}$$

$$= \frac{1}{3}\left\{(x^3+1)\log(x+1) - \int (x^2-x+1)dx\right\}$$

$$= \frac{1}{3}\left\{(x^3+1)\log(x+1) - \frac{1}{3}x^3 + \frac{1}{2}x^2 - x\right\} + C$$

$$= -\frac{1}{9}x^3 + \frac{1}{6}x^2 - \frac{1}{3}x + \frac{1}{3}(x^3+1)\log(x+1) + C$$

解法 2

$x+1=t$ とおいて地道に解くと,

$$\int x^2 \log(x+1)dx = \int (t-1)^2 \log t dt$$

$$= \int \left\{\frac{(t-1)^3}{3}\right\}' \log t dt$$

$$= \frac{1}{3}\left\{(t-1)^3 \log t - \int (t-1)^3 (\log t)' dt\right\}$$

$$= \frac{1}{3}\left\{x^3 \log(x+1) - \int \frac{(t-1)^3}{t}dt\right\}$$

$$= \frac{1}{3}\left\{x^3 \log(x+1) - \int \left(t^2 - 3t + 3 - \frac{1}{t}\right)dt\right\}$$

$$= \frac{1}{3}\left\{x^3 \log(x+1) - \frac{1}{3}t^3 + \frac{3}{2}t^2 - 3t + \log t\right\} + C'$$

$$= \frac{1}{3}\left\{\{x^3+1\}\log(x+1) - \frac{1}{3}(x+1)^3 + \frac{3}{2}(x+1)^2 - 3(x+1)\right\} + C'$$

$$= \frac{1}{3}(x^3+1)\log(x+1) - \frac{1}{9}x^3 + \frac{1}{6}x^2 - \frac{1}{3}x + C$$

(4)

$e^x=t$ とおくと，$e^x=\dfrac{dt}{dx}$ より，$dx=\dfrac{dt}{e^x}=\dfrac{dt}{t}$

よって，

$$\int\frac{1}{1-e^x}dx=\int\frac{1}{1-t}\cdot\frac{dt}{t}$$
$$=\int\frac{-1}{t(t-1)}dt$$
$$=\int\left(\frac{1}{t}-\frac{1}{t-1}\right)dt$$
$$=\log|t|-\log|t-1|+C$$
$$=\log\left|\frac{t}{t-1}\right|+C$$
$$=\log\frac{e^x}{|e^x-1|}+C$$

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(1)

$$\int x^n\sin xdx=\int x^n(-\cos x)'dx$$
$$=-x^n\cos x+\int(x^n)'\cos xdx$$
$$=-x^n\cos x+n\int x^{n-1}\cos xdx$$

(2)

$\displaystyle\int\tan^n xdx=\frac{1}{n-1}\tan^{n-1}x-\int\tan^{n-2}xdx$ の証明

$$\int\tan^n xdx=\int\tan^2 x\cdot\tan^{n-2}xdx$$
$$=\int\left(\frac{1}{\cos^2 x}-1\right)\tan^{n-2}xdx$$
$$=\int\left\{(\tan x)'-1\right\}\tan^{n-2}xdx$$
$$=\int\left\{(\tan x)'\tan^{n-2}x-\tan^{n-2}x\right\}dx$$
$$=\int(\tan x)'\tan^{n-2}xdx-\int\tan^{n-2}xdx$$
$$=\frac{1}{n-1}\tan^{n-1}x-\int\tan^{n-2}xdx$$

$\int \tan^4 x dx$

$\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx$ （$n \geq 2$） より，

$\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \int \tan^2 x dx$ ・・・①

$\int \tan^2 x dx = \tan x - \int dx = \tan x - x + C'$ ・・・②

①，②より，

$\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C$