積分法 5　定積分とその基本性質

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(1)

$$\int_0^1 \sqrt{e^{1-t}}\,dt = \int_0^1 e^{\frac{1-t}{2}}\,dt$$
$$= \left[-2e^{\frac{1-t}{2}}\right]_0^1$$
$$= -2\left[e^{\frac{1-t}{2}}\right]_0^1$$
$$= -2\left(e^0 - e^{\frac{1}{2}}\right)$$
$$= -2\left(1 - e^{\frac{1}{2}}\right)$$
$$= -2\left(1 - \sqrt{e}\right)$$
$$= 2\left(\sqrt{e} - 1\right)$$

(2)

$$\frac{\cos 2\theta}{\sin\theta + \cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta + \sin\theta}$$
$$= \frac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{\cos\theta + \sin\theta}$$
$$= \cos\theta - \sin\theta$$

より,

$$\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta}{\sin\theta + \cos\theta}\,d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta - \sin\theta)\,d\theta$$
$$= \left[\sin\theta + \cos\theta\right]_0^{\frac{\pi}{2}}$$
$$= \sin\frac{\pi}{2} + \cos\frac{\pi}{2} - (\sin 0 + \cos 0)$$
$$= 1 + 0 - (0 + 1)$$
$$= 1 - 1$$
$$= 0$$

(3)

$$\begin{aligned}\sin^4 x &= (\sin^2 x)^2\\ &= \left(\frac{1-\cos 2x}{2}\right)^2\\ &= \frac{1}{4}(1-2\cos 2x+\cos^2 2x)\\ &= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x\\ &= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)\\ &= \frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x+\frac{3}{8}\end{aligned}$$

より,

$$\begin{aligned}\int_0^\pi \sin^4 x dx &= \int_0^\pi \left(\frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x+\frac{3}{8}\right)dx\\ &= \left[\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+\frac{3}{8}x\right]_0^\pi\\ &= \frac{3}{8}\pi\end{aligned}$$

(4)

$$\begin{aligned}\frac{\sqrt{x^2-4x+4}}{x} &= \frac{\sqrt{(x-2)^2}}{x}\\ &= \frac{|x-2|}{x}\end{aligned}$$

より,

$$\begin{aligned}\int_1^2 \frac{\sqrt{x^2-4x+4}}{x}dx &= \int_1^2 \frac{|x-2|}{x}dx\\ &= \int_1^2 \frac{-(x-2)}{x}dx\\ &= \int_1^2 \left(-1+\frac{2}{x}\right)dx\\ &= [-x+2\log x]_1^2\\ &= -2+2\log 2-(-1+2\log 1)\\ &= -1+2\log 2\end{aligned}$$

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(1)

$0 \leq 2\theta \leq 2\pi$ より，

$0 \leq 2\theta \leq \dfrac{\pi}{2},\ \dfrac{3}{2}\pi \leq 2\theta \leq 2\pi$，すなわち $0 \leq \theta \leq \dfrac{\pi}{4},\ \dfrac{3}{4}\pi \leq \theta \leq \pi$ のとき，$\cos 2\theta \geq 0$

$\dfrac{\pi}{2} \leq 2\theta \leq \dfrac{3}{2}\pi$，すなわち $\dfrac{\pi}{4} \leq \theta \leq \dfrac{3}{4}\pi$ のとき，$\cos 2\theta \leq 0$

よって，

$$\begin{aligned}\int_0^{\pi} |\cos 2\theta| d\theta &= \int_0^{\frac{\pi}{4}} \cos 2\theta + \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} (-\cos 2\theta) d\theta + \int_{\frac{3}{4}\pi}^{\pi} \cos 2\theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \cos 2\theta + \int_{\frac{3}{4}\pi}^{\frac{\pi}{4}} \cos 2\theta d\theta + \int_{\frac{3}{4}\pi}^{\pi} \cos 2\theta d\theta \\ &= \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{2}\sin 2\theta\right]_{\frac{3}{4}\pi}^{\frac{\pi}{4}} + \left[\frac{1}{2}\sin 2\theta\right]_{\frac{3}{4}\pi}^{\pi} \\ &= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{2} \\ &= 2\end{aligned}$$

(2)

$\sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$，$\dfrac{\pi}{4} \leq x + \dfrac{\pi}{2} \leq \pi + \dfrac{\pi}{4}$ より，

$\dfrac{\pi}{4} \leq x + \dfrac{\pi}{4} \leq \pi$，すなわち $0 \leq x \leq \dfrac{3}{4}\pi$ のとき $\sin x + \cos x \geq 0$

$\pi \leq x + \dfrac{\pi}{4} \leq \pi + \dfrac{\pi}{4}$，すなわち $\dfrac{3}{4}\pi \leq x \leq \pi$ のとき $\sin x + \cos x \leq 0$

よって，

$$\begin{aligned}\int_0^{\pi} |\sin x + \cos x| dx &= \int_0^{\frac{3}{4}\pi} (\sin x + \cos x) dx + \int_{\frac{3}{4}\pi}^{\pi} \{-(\sin x + \cos x)\} dx \\ &= \int_0^{\frac{3}{4}\pi} (\sin x + \cos x) dx + \int_{\pi}^{\frac{3}{4}\pi} (\sin x + \cos x) dx \\ &= [-\cos x + \sin x]_0^{\frac{3}{4}\pi} + [-\cos x + \sin x]_{\pi}^{\frac{3}{4}\pi} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) \\ &= 2\sqrt{2}\end{aligned}$$

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(1)

$m=n$ のとき

$$\cos mx\cos nx=\cos^2 mx$$
$$=\frac{1+\cos 2mx}{2}$$

より,

$$\int_0^{\pi}\cos mx\cos nxdx=\frac{1}{2}\int_0^{\pi}(1+\cos 2mx)dx$$
$$=\frac{1}{2}\left[x+\frac{1}{4}\sin 2mx\right]_0^{\pi}$$
$$=\frac{1}{2}\pi$$

$m\neq n$ のとき

$\cos mx\cos nx=\frac{1}{2}\{\cos(m+n)x+\cos(m-n)x\}$ より,

$$\int_0^{\pi}\cos mx\cos nxdx=\frac{1}{2}\int_0^{\pi}\{\cos(m+n)x+\cos(m-n)x\}dx$$
$$=\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right]_0^{\pi}$$
$$=0$$

(2)

$m=n$ のとき

$$\sin mx\sin nx=\sin^2 mx$$
$$=\frac{1-\cos 2mx}{2}$$

より,

$$\int_0^{\pi}\sin mx\sin nxdx=\frac{1}{2}\int_0^{\pi}(1-\cos 2mx)dx$$
$$=\frac{1}{2}\left[x-\frac{1}{4}\sin 2mx\right]_0^{\pi}$$
$$=\frac{1}{2}\pi$$

$m \neq n$ のとき

$\sin mx \sin nx = \dfrac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$ より,

$$\begin{aligned}\int_0^{\pi} \sin mx \sin nx dx &= \frac{1}{2}\int_0^{\pi}\{\cos(m-n)x - \cos(m+n)x\}dx \\ &= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi} \\ &= 0\end{aligned}$$

(3)

$m = n$ のとき

$$\begin{aligned}\sin mx \cos nx &= \sin mx \cos mx \\ &= \frac{\sin 2mx}{2}\end{aligned}$$

より,

$$\begin{aligned}\int_0^{\pi} \sin mx \sin nx dx &= \frac{1}{2}\int_0^{\pi} \sin 2mx dx \\ &= \frac{1}{2}\left[-\frac{\cos 2mx}{2}\right]_0^{\pi} \\ &= 0\end{aligned}$$

$m \neq n$ のとき

$\sin mx \cos nx = \dfrac{1}{2}\{\sin(m+n)x + \sin(m-n)x\}$ より,

$$\begin{aligned}\int_0^{\pi} \sin mx \cos nx dx &= \frac{1}{2}\int_0^{\pi}\{\sin(m+n)x + \sin(m-n)x\}dx \\ &= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_0^{\pi} \\ &= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_0^{\pi} \\ &= -\frac{1}{2}\left\{\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} - \left(\frac{1}{m+n} + \frac{1}{m-n}\right)\right\}\end{aligned}$$

よって,

m と n の遇奇が一致するとき

$m+n$ と $m-n$ は偶数だから,

$$\begin{aligned}\int_0^{\pi} \sin mx \cos nx dx &= -\frac{1}{2}\left\{\frac{1}{m+n} + \frac{1}{m-n} - \left(\frac{1}{m+n} + \frac{1}{m-n}\right)\right\} \\ &= 0\end{aligned}$$

m と n の偶奇が一致しないとき

$m+n$ と $m-n$ は偶数だから，

$$\int_0^{\pi}\sin mx\cos nxdx=-\frac{1}{2}\left\{-\frac{1}{m+n}-\frac{1}{m-n}-\left(\frac{1}{m+n}+\frac{1}{m-n}\right)\right\}$$
$$=\frac{2m}{m^2-n^2}$$

$m=n$ の場合と $m\neq n$ の偶奇が一致する場合で定積分の値が等しいことから，
以上をまとめると

m と n の偶奇が一致するとき：$\int_0^{\pi}\sin mx\cos nxdx=0$

m と n の偶奇が一致しないとき：$\int_0^{\pi}\sin mx\cos nxdx=\dfrac{2m}{m^2-n^2}$

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$$(1-a\sin x-b\sin 2x)^2=1+a^2\sin^2 x+b^2\sin^2 2x-2a\sin x+2ab\sin x\sin 2x-2b\sin 2x$$
$$=1+a^2\cdot\frac{1-\cos 2x}{2}+b^2\cdot\frac{1-\cos 4x}{2}$$
$$-2a\sin x+2ab\cdot\frac{\cos(2x-x)-\cos(2x+x)}{2}-2b\sin 2x$$
$$=-\frac{a^2\cos 2x}{2}-\frac{b^2\cos 4x}{2}+ab\cos x-ab\cos 3x-2a\sin x-2b\sin 2x+\frac{a^2+b^2+2}{2}$$

$$\int_0^{\pi}(1-axinx-b\sin 2x)^2dx$$
$$=\int_0^{\pi}\left(-\frac{a^2\cos 2x}{2}-\frac{b^2\cos 4x}{2}+ab\cos x-ab\cos 3x-2a\sin x-2b\sin 2x+\frac{a^2+b^2+2}{2}\right)dx$$
$$=\int_0^{\pi}\left(-2a\sin x-2b\sin 2x+\frac{a^2+b^2+2}{2}\right)dx$$
$$=\left[2a\cos x+b\cos 2x+\frac{a^2+b^2+2}{2}x\right]_0^{\pi}$$
$$=-2a+b+\frac{a^2+b^2+2}{2}\pi-2a-b$$
$$=\frac{\pi}{2}a^2-4a+\frac{\pi}{2}b^2+\pi$$
$$=\frac{\pi}{2}\left(a-\frac{4}{\pi}\right)^2+\frac{\pi}{2}b^2+\pi-\frac{8}{\pi}$$

よって，$a=\dfrac{4}{\pi}$, $b=0$ のとき最小となる。