積分法 6　定積分の置換積分法

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(1)

$\sqrt{3x+4}=t$ とおくと，$3x+4=t^2$ より，$x=\dfrac{t^2-4}{3},\ dx=\dfrac{2t}{3}dt$

また，$x=0\Rightarrow t=2,\ x=-1\Rightarrow t=1$

よって，

$$\int_{-1}^{0}(x+2)\sqrt{3x+4}dx=\int_{1}^{2}\left(\frac{t^2-4}{3}+2\right)\cdot t\cdot\frac{2t}{3}dt$$
$$=\frac{2}{9}\int_{1}^{2}\left(t^4+2t^2\right)dt$$
$$=\frac{2}{9}\left[\frac{1}{5}t^5+\frac{2}{3}t^3\right]_1^2$$
$$=\frac{2}{9}\left(\frac{31}{5}+\frac{14}{3}\right)$$
$$=\frac{326}{135}$$

(2)

$\sqrt{x+1}=t$ とおくと，$x+1=t^2$ より，$x=t^2-1,\ dx=2tdt$

また，$x=4\Rightarrow t=\sqrt{5},\ x=0\Rightarrow t=1$

よって，

$$\int_{0}^{4}\frac{x^2}{\sqrt{x+1}}dx=\int_{1}^{\sqrt{5}}\frac{\left(t^2-1\right)^2}{t}\cdot 2tdt$$
$$=2\int_{1}^{\sqrt{5}}\left(t^4-2t^2+1\right)dt$$
$$=2\left[\frac{1}{5}t^5-\frac{2}{3}t^3+t\right]_1^{\sqrt{5}}$$
$$=\frac{16}{15}\left(5\sqrt{5}-1\right)$$

(3)

$\sqrt{1+x^2}=t$ とおくと， $1+x^2=t^2$ より， $x^2=t^2-1,\ xdx=tdt$

また， $x=1 \Rightarrow t=\sqrt{2},\ x=0 \Rightarrow t=1$

よって，

$$\int_0^1 \frac{x^3}{\sqrt{1+x^2}}dx=\int_0^1 \frac{x^2}{\sqrt{1+x^2}}\cdot xdx$$
$$=\int_1^{\sqrt{2}} \frac{t^2-1}{t}\cdot tdt$$
$$=\int_1^{\sqrt{2}} \left(t^2-1\right)dt$$
$$=\left[\frac{1}{3}t^3-t\right]_1^{\sqrt{2}}$$
$$=\frac{2-\sqrt{2}}{3}$$

(4)

$\sqrt{x+1}=t$ とおくと， $x+1=t^2$ より， $x=t^2-1,\ dx=2tdt$

また， $x=3 \Rightarrow t=2,\ x=1 \Rightarrow t=\sqrt{2}$

よって，

$$\int_1^3 \frac{dx}{x\sqrt{x+1}}=\int_{\sqrt{2}}^2 \frac{2tdt}{\left(t^2-1\right)t}$$
$$=\int_{\sqrt{2}}^2 \frac{2}{(t-1)(t+1)}dt$$
$$=\int_{\sqrt{2}}^2 \left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$
$$=\left[\log|t-1|-\log|t+1|\right]_{\sqrt{2}}^2$$
$$=\left[\log\left|\frac{t-1}{t+1}\right|\right]_{\sqrt{2}}^2$$
$$=\log\frac{1}{3}-\log\frac{\sqrt{2}-1}{\sqrt{2}+1}$$
$$=\log\frac{\sqrt{2}+1}{3\left(\sqrt{2}-1\right)}$$
$$=\log\frac{3+2\sqrt{2}}{3}$$

(5)

$e^x-1=t$ とおくと，$e^x=t+1$ より，$x=\log|t+1|,\ dx=\dfrac{dt}{t+1}$

また，$x=2\Rightarrow t=e^2-1,\ x=1\Rightarrow t=e-1$

よって，

$$\begin{aligned}\int_1^2\frac{dx}{e^x-1}&=\int_{e-1}^{e^2-1}\frac{1}{t}\cdot\frac{dt}{t+1}\\&=\int_{e-1}^{e^2-1}\left(\frac{1}{t}-\frac{1}{t+1}\right)dt\\&=\left[\log|t|-\log|t+1|\right]_{e-1}^{e^2-1}\\&=\left[\log\left|\frac{t}{t+1}\right|\right]_{e-1}^{e^2-1}\\&=\log\frac{e^2-1}{e^2}-\log\frac{e-1}{e}\\&=\log\frac{(e^2-1)e}{e^2(e-1)}\\&=\log\frac{e+1}{e}\end{aligned}$$

(6)

$\cos x=t$ とおくと $dx=-\dfrac{dt}{\sin x}$，$x=\dfrac{\pi}{4}\Rightarrow t=\dfrac{\sqrt{2}}{2},\ x=0\Rightarrow t=1$

より，

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\frac{\sin^3 x}{\cos^2 x}dx&=\int_1^{\frac{\sqrt{2}}{2}}\frac{\sin^3 x}{t^2}\cdot\left(-\frac{1}{\sin x}\right)dt\\&=-\int_1^{\frac{\sqrt{2}}{2}}\frac{\sin^2 x}{t^2}dt\\&=-\int_1^{\frac{\sqrt{2}}{2}}\frac{1-\cos^2 x}{t^2}dt\\&=\int_1^{\frac{\sqrt{2}}{2}}\frac{t^2-1}{t^2}dt\\&=\int_1^{\frac{\sqrt{2}}{2}}\left(1-t^{-2}\right)dt\\&=\left[t+\frac{1}{t}\right]_1^{\frac{\sqrt{2}}{2}}\\&=\frac{3\sqrt{2}}{2}-2\end{aligned}$$

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(1)

$\sqrt{2x-x^2}=\sqrt{1-(x-1)^2}$

ここで，$x-1=\sin\theta$ （$-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$）とおくと，

$\sqrt{1-(x-1)^2}=\sqrt{1-\sin^2\theta}=|\cos\theta|=\cos\theta\left(\because -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$, $dx=\cos\theta\,d\theta$

$x=1\Rightarrow\sin\theta=0$ より $\theta=0$，$x=0\Rightarrow\sin\theta=-1$ より $\theta=-\frac{\pi}{2}$

$$\begin{aligned}\therefore\int_0^1\sqrt{2x-x^2}\,dx&=\int_{-\frac{\pi}{2}}^{0}\cos\theta\cdot\cos\theta\,d\theta\\&=\int_{-\frac{\pi}{2}}^{0}\cos^2\theta\,d\theta\\&=\int_{-\frac{\pi}{2}}^{0}\frac{1+\cos 2\theta}{2}d\theta\\&=\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{0}\\&=\frac{\pi}{4}\end{aligned}$$

(2)

$\frac{1}{\sqrt{2x-x^2}}=\frac{1}{\sqrt{1-(x-1)^2}}$

ここで，$x-1=\sin\theta$ （$-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$）とおくと，

$\frac{1}{\sqrt{1-(x-1)^2}}=\frac{1}{\sqrt{1-\sin^2\theta}}=\frac{1}{\sqrt{\cos^2\theta}}=\frac{1}{|\cos\theta|}=\frac{1}{\cos\theta}$, $dx=\cos\theta\,d\theta$

$x=\frac{1}{2}\Rightarrow\sin\theta=-\frac{1}{2}$ より $\theta=-\frac{\pi}{6}$，$x=1\Rightarrow\sin\theta=0$ より $\theta=0$

$$\begin{aligned}\therefore\int_1^{\frac{1}{2}}\frac{dx}{\sqrt{2x-x^2}}dx&=\int_0^{-\frac{\pi}{6}}\frac{\cos\theta\,d\theta}{\cos\theta}\\&=[\theta]_0^{-\frac{\pi}{6}}\\&=-\frac{\pi}{6}\end{aligned}$$

(3)

$x+1=\tan\theta\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$ とおくと，$dx=\frac{d\theta}{\cos^2\theta}$

$x=0\Rightarrow\tan\theta=1$ より $\theta=\frac{\pi}{4}$，$x=-1\Rightarrow\tan\theta=0$ より $\theta=0$

$$\therefore\int_{-1}^{0}\frac{dx}{(x+1)^2+1}=\int_0^{\frac{\pi}{4}}\frac{1}{\tan^2\theta+1}\cdot\frac{d\theta}{\cos^2\theta}$$
$$=\int_0^{\frac{\pi}{4}}\frac{1}{\frac{1}{\cos^2\theta}}\cdot\frac{d\theta}{\cos^2\theta}$$
$$=\int_0^{\frac{\pi}{4}}d\theta$$
$$=[\theta]_0^{\frac{\pi}{4}}$$
$$=\frac{\pi}{4}$$

(4)

$$\frac{1}{x^2-2x+2}=\frac{1}{(x-1)^2+1}$$

ここで，$x-1=\tan\theta\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$ とおくと，$dx=\frac{d\theta}{\cos^2\theta}$

$x=2\Rightarrow\tan\theta=1$ より $\theta=\frac{\pi}{4}$，$x=-1\Rightarrow\tan\theta=0$ より $\theta=0$

$$\therefore\int_1^2\frac{dx}{x^2-2x+2}=\int_0^{\frac{\pi}{4}}\frac{1}{\tan^2\theta+1}\cdot\frac{d\theta}{\cos^2\theta}$$
$$=\int_0^{\frac{\pi}{4}}d\theta$$
$$=[\theta]_0^{\frac{\pi}{4}}$$
$$=\frac{\pi}{4}$$

(5)

$$\int_1^{\sqrt{3}} \frac{2x+1}{x^2+1}dx = \int_1^{\sqrt{3}} \frac{2x}{x^2+1}dx + \int_1^{\sqrt{3}} \frac{1}{x^2+1}dx$$

ここで,

$\displaystyle\int_1^{\sqrt{3}} \frac{2x}{x^2+1}dx$ について

$$\begin{aligned}\int_1^{\sqrt{3}} \frac{2x}{x^2+1}dx &= \int_1^{\sqrt{3}} \frac{\left(x^2+1\right)'}{x^2+1}dx \\ &= \left[\log\left(x^2+1\right)\right]_1^{\sqrt{3}} \\ &= \log 4 - \log 2 \\ &= \log 2\end{aligned}$$

$\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2+1}dx$ について

$x = \tan\theta \left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$ とおくと, $dx = \dfrac{d\theta}{\cos^2\theta}$

$x = \sqrt{3} \Rightarrow \tan\theta = \sqrt{3}$ より $\theta = \dfrac{\pi}{3}$, $x = 1 \Rightarrow \tan\theta = 1$ より $\theta = \dfrac{\pi}{4}$

$$\begin{aligned}\therefore \int_1^{\sqrt{3}} \frac{1}{x^2+1}dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta} \cdot \frac{d\theta}{\cos^2\theta} \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta \\ &= \left[\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \frac{\pi}{12}\end{aligned}$$

よって, $\displaystyle\int_1^{\sqrt{3}} \frac{2x+1}{x^2+1}dx = \log 2 + \frac{\pi}{12}$

(6)

$x = a\tan\theta \left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$ とおくと，$dx = \dfrac{a}{\cos^2\theta}d\theta$

$x = a \Rightarrow \tan\theta = 1$ より $\theta = \dfrac{\pi}{4}$，$x = 0 \Rightarrow \tan\theta = 0$ より $\theta = 0$

$$\begin{aligned}\therefore \int_0^a \frac{dx}{\left(x^2+a^2\right)^2} &= \int_0^{\frac{\pi}{4}} \frac{1}{a^4\left(\tan^2\theta+1\right)^2}\frac{a}{\cos^2\theta}d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{\cos^2\theta}{a^3}d\theta \\ &= \frac{1}{2a^3}\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)d\theta \\ &= \frac{1}{2a^3}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2a^3}\left(\frac{\pi}{4}+\frac{1}{2}\right) \\ &= \frac{\pi+2}{8a^3}\end{aligned}$$

(7)

解法 **1**

$2 - x^2 = t$ とおくと，$dx = -\dfrac{dt}{2x}$

$x = 1 \Rightarrow t = 1$，$x = 0 \Rightarrow t = 2$

$$\begin{aligned}\therefore \int_0^1 \frac{x}{\left(2-x^2\right)^2}dx &= \int_2^1 \frac{x}{t^2}\cdot\left(-\frac{1}{2x}\right)dt \\ &= \frac{1}{2}\int_2^1 -t^{-2}dt \\ &= \frac{1}{2}\left[\frac{1}{t}\right]_2^1 \\ &= \frac{1}{4}\end{aligned}$$

解法 2

$x=\sqrt{2}\sin\theta\left(-\frac{\pi}{2}\leq\theta\leq\frac{\theta}{2}\right)$ とおくと， $dx=\sqrt{2}\cos\theta\,d\theta$

$x=1\Rightarrow\theta=\frac{\pi}{4}$， $x=0\Rightarrow\theta=0$

$$\begin{aligned}\therefore\int_0^1\frac{x}{\left(2-x^2\right)^2}dx&=\int_0^{\frac{\pi}{4}}\frac{\sqrt{2}\sin\theta}{\left\{2\left(1-\sin^2\theta\right)\right\}^2}\cdot\sqrt{2}\cos\theta\,d\theta\\&=\int_0^{\frac{\pi}{4}}\frac{\sqrt{2}\sin\theta}{4\cos^4\theta}\cdot\sqrt{2}\cos\theta\,d\theta\\&=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}\cdot\frac{\sin\theta}{\cos\theta}d\theta\\&=\frac{1}{2}\int_0^{\frac{\pi}{4}}(\tan\theta)'\tan\theta\,d\theta\\&=\frac{1}{4}\left[\tan^2\theta\right]_0^{\frac{\pi}{4}}\\&=\frac{1}{4}\end{aligned}$$

(8)

$x=a\sin\theta\left(-\frac{\pi}{2}\leq\theta\leq\frac{\theta}{2}\right)$ とおくと， $dx=a\cos\theta\,d\theta$

$x=\frac{a}{2}\Rightarrow\theta=\frac{\pi}{6}$， $x=0\Rightarrow\theta=0$

$$\begin{aligned}\therefore\int_0^{\frac{a}{2}}\frac{dx}{\left(a^2-x^2\right)^{\frac{3}{2}}}dx&=\int_0^{\frac{\pi}{6}}\frac{a\cos\theta d\theta}{\left\{a^2\left(1-\sin^2\theta\right)\right\}^{\frac{3}{2}}}\\&=\frac{1}{a^2}\int_0^{\frac{\pi}{6}}\frac{1}{\cos^2\theta}d\theta\\&=\frac{1}{a^2}\left[\tan\theta\right]_0^{\frac{\pi}{6}}\\&=\frac{\sqrt{3}}{3a^2}\end{aligned}$$

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(1)

$a+b-x=t$ とおくと，$dx=-dt$，$x=b\Rightarrow t=a$，$x=a\Rightarrow t=b$

$\therefore \int_a^b f(a+b-x)dx=-\int_b^a f(t)dt=\int_a^b f(t)dt$

これと $\int_a^b f(t)dt=\int_a^b f(x)dx$ より，$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

(2)

$-x=t$ とおくと，$dx=-dt$，$x=a\Rightarrow t=-a$，$x=0\Rightarrow t=0$

$\therefore \int_0^a f(-x)dx=-\int_0^{-a} f(t)dt=\int_{-a}^0 f(t)dt$

これと $\int_{-a}^0 f(t)dt=\int_{-a}^0 f(x)dx$ より，$\int_{-a}^0 f(x)dx=\int_0^a f(-x)dx$

よって，

$$\begin{aligned}\int_{-a}^a f(x)dx&=\int_0^a f(x)dx+\int_{-a}^0 f(x)dx\\&=\int_0^a f(x)dx+\int_0^a f(-x)dx\\&=\int_0^a \{f(x)+f(-x)\}dx\end{aligned}$$

(3)

解法 1

$a-x=t$ とおくと，$dx=-dt$，$x=\frac{a}{2}\Rightarrow t=\frac{a}{2}$，$x=0\Rightarrow t=a$

$\therefore \int_0^{\frac{a}{2}} f(a-x)dx=-\int_a^{\frac{a}{2}} f(t)dt=\int_{\frac{a}{2}}^a f(t)dt$

これと $\int_{\frac{a}{2}}^a f(t)dt=\int_{\frac{a}{2}}^a f(x)dx$ より，$\int_{\frac{a}{2}}^a f(x)dx=\int_0^{\frac{a}{2}} f(a-x)dx$

よって，

$$\begin{aligned}\int_0^a f(x)dx&=\int_{\frac{a}{2}}^a f(x)dx+\int_0^{\frac{a}{2}} f(x)dx\\&=\int_0^{\frac{a}{2}} f(a-x)dx+\int_0^{\frac{a}{2}} f(x)dx\\&=\int_0^{\frac{a}{2}} \{f(x)+f(a-x)\}dx\end{aligned}$$

解法 2： $x=\frac{a}{2}$ が積分区間の中線であることと対称性を利用

$y=f(x)$ 上の任意の点 $(x,\ f(x))$ を $x=\frac{a}{2}$ に関して対称移動した点を $(X,\ Y)$ とすると，

$\frac{x+X}{2}=\frac{a}{2}$， $Y=f(x)$ より， $Y=f(a-X)$

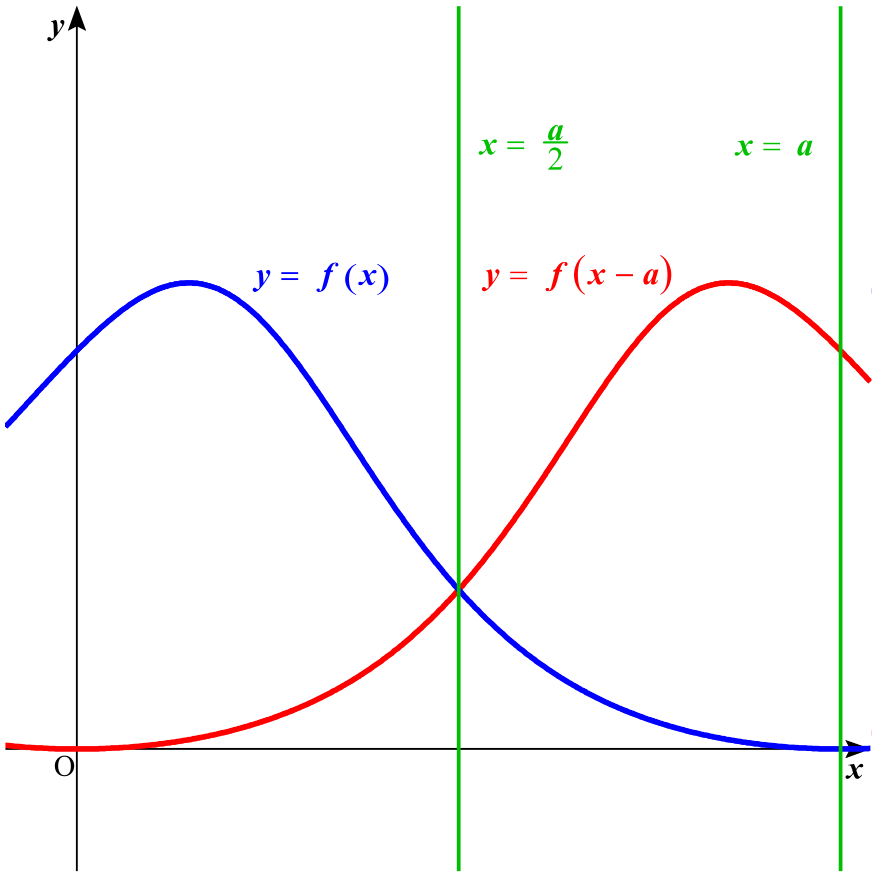
よって， $y=f(x)$ を $x=\frac{a}{2}$ に関して対称移動した関数は $y=f(x-a)$ であり，

これより， $\int_{\frac{a}{2}}^{a} f(x-a)dx=\int_{0}^{\frac{a}{2}} f(x)dx$ が成り立つ。

ゆえに，

$$\begin{aligned}\int_0^a f(x)dx &= \int_{\frac{a}{2}}^{a} f(x)dx+\int_0^{\frac{a}{2}} f(x)dx\\ &= \int_0^{\frac{a}{2}} f(a-x)dx+\int_0^{\frac{a}{2}} f(x)dx\\ &= \int_0^{\frac{a}{2}} \{f(x)+f(a-x)\}dx\end{aligned}$$

参考図



(4)

$y=f(x)$とすると，$f(a+x)=f(a-x)$より，

$y=f(x)$のグラフは$x=a$に関して対称であるから，与式が成り立つのは明らかである。

具体的には，

$\int_{a-b}^{a+b}f(x)dx=\int_{a}^{a+b}f(x)dx+\int_{a-b}^{a}f(x)dx$　・・・①　だから，

$\int_{a}^{a+b}f(x)dx=\int_{a-b}^{a}f(x)dx$が成り立つことを示せばよく，

それには，おそらく条件$f(a+x)=f(a-x)$を使う必要があるだろう。

この関係を使うには，$y=f(x)$をx軸方向に$-a$平行移動すればよい。

つまり，$y=f(x)$をx軸方向に$-a$平行移動したグラフは$y=f(x+a)$であるから，

$f(a+x)=f(a-x)$の関係が使える。

$y=f(x)$をx軸方向に$-a$平行移動したグラフは$y=f(x+a)$であり，

積分区間は$[a,\ a-b]$から$[0,\ -b]$に移される。

したがって，$\int_{a-b}^{a}f(x)dx=\int_{-b}^{0}f(x+a)dx$が成り立つ。

これと$f(a+x)=f(a-x)$より，$\int_{a-b}^{a}f(x)dx=\int_{-b}^{0}f(a-x)dx$　・・・②

$\int_{-b}^{0}f(a-x)dx$の置換積分については，

$a-x=t$とおくと，$dx=-dt$，$x=0\Rightarrow t=a$，$x=-b\Rightarrow t=a+b$

よって，$\int_{-b}^{0}f(a-x)dx=-\int_{a+b}^{a}f(t)dt=\int_{a}^{a+b}f(t)dt$

$\int_{a}^{a+b}f(t)dt=\int_{a}^{a+b}f(x)dx$であるから，これと②より，

$\int_{a-b}^{a}f(x)dx=\int_{b}^{a+b}f(x)dx$

ゆえに，①から，$\int_{a-b}^{a+b}f(x)dx=\int_{a}^{a+b}f(x)dx+\int_{a}^{a+b}f(x)dx=2\int_{a}^{a+b}f(x)dx$となる。

まとめると,

$\int_{a-b}^{a+b} f(x)dx = \int_{a}^{a+b} f(x)dx + \int_{a-b}^{a} f(x)dx$ ・・・①

$\int_{a-b}^{a} f(x)dx$ について

$x-a=t$ とおくと, $dx=dt$, $x=a \Rightarrow t=0$, $x=a-b \Rightarrow t=-b$ より,

$\int_{a-b}^{a} f(x)dx = \int_{-b}^{0} f(a+t)dx$

これと条件より, $f(a+t)=f(a-t)$ だから, $\int_{-b}^{0} f(a+t)dx = \int_{-b}^{0} f(a-t)dt$

よって, $\int_{a-b}^{a} f(x)dx = \int_{-b}^{0} f(a-t)dt$ ・・・②

$\int_{-b}^{0} f(a-t)dt$ について

$a-t=u$ とおくと, $dt=-du$, $t=0 \Rightarrow u=a$, $t=-b \Rightarrow u=a+b$ より,

$\int_{-b}^{0} f(a-t)dt = -\int_{a+b}^{a} f(u)du = \int_{a}^{a+b} f(u)du$

$\int_{a}^{a+b} f(u)du = \int_{a}^{a+b} f(x)dx$ だから, $\int_{-b}^{0} f(a-t)dt = \int_{a}^{a+b} f(x)dx$ ・・・③

①,②,③より,

$\int_{a-b}^{a+b} f(x)dx = \int_{a}^{a+b} f(x)dx + \int_{a}^{a+b} f(x)dx = 2\int_{a}^{a+b} f(x)dx$

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$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

解法 1： $\sin x = \sin(\pi - x)$ を利用

$x = \pi - t$ とおくと， $dx = -dt$， $x = \pi \Rightarrow t = 0$， $x = 0 \Rightarrow t = \pi$ より，

$$\begin{aligned}\int_0^{\pi} xf(\sin x)dx &= -\int_{\pi}^{0} (\pi - t)f(\sin(\pi - t))dt \\ &= -\pi\int_{\pi}^{0} f(\sin t)dt + \int_{\pi}^{0} tf(\sin t)dt \\ &= \pi\int_0^{\pi} f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt\end{aligned}$$

これと $\pi\int_0^{\pi} f(\sin t)dt = \pi\int_0^{\pi} f(\sin x)dx$， $\int_0^{\pi} tf(\sin t)dt = \int_0^{\pi} xf(\sin x)dx$ より，

$$\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\pi} f(\sin x)dx - \int_0^{\pi} xf(\sin x)dx$$

よって， $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

解法 2：奇関数の性質を利用

$$\int_0^{\pi} xf(\sin x)dx - \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx = \int_0^{\pi}\left(x - \frac{\pi}{2}\right)f(\sin x)dx$$

ここで， $x - \frac{\pi}{2} = t$ とおくと， $dx = dt$， $x = \pi \Rightarrow t = \frac{\pi}{2}$， $x = 0 \Rightarrow t = -\frac{\pi}{2}$ より，

$$\int_0^{\pi}\left(x - \frac{\pi}{2}\right)f(\sin x)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf\left(\sin\left(t + \frac{\pi}{2}\right)\right)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf(\cos t)dt$$

$y = t$ は奇関数， $y = f(\cos t)$ は， $f(\cos t) = f(\cos(-t))$ より， $y = f(\cos t)$ は偶関数だから，
$tf(\cos t)$ は奇関数である。

よって， $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf(\cos t)dt = 0$

ゆえに， $\int_0^{\pi} xf(\sin x)dx - \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx = 0$

すなわち， $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

$\dfrac{\sin x}{1+\cos^2 x}=\dfrac{\sin x}{2-\sin^2 x}$ より，$f(\sin x)=\dfrac{\sin x}{2-\sin^2 x}$ とすると，

$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\int_0^{\pi}xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi}f(\sin x)dx=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

ここで，

$\cos x=\tan\theta\left(-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}\right)$ とおくと，

$dx=-\dfrac{d\theta}{\sin x\cos^2\theta}$，　$x=\pi\Rightarrow\theta=-\dfrac{\pi}{4}$，　$x=0\Rightarrow\theta=\dfrac{\pi}{4}$ より，

$$\begin{aligned}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx&=-\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\frac{\sin x}{1+\tan^2\theta}\cdot\frac{d\theta}{\sin x\cos^2\theta}\\&=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta\\&=[\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\\&=\frac{\pi}{2}\end{aligned}$$

よって，$\displaystyle\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\frac{\pi^2}{4}$

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条件より，

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx=\int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)=\sin\left(\frac{\pi}{2}-x\right)}dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$$

また，

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx+\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{2}} dx$$

$$=\left[x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}$$

よって，$2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx=\frac{\pi}{2}$

ゆえに，$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx=\frac{\pi}{4}$

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$$\frac{1}{x^3+8}=\frac{1}{(x+2)(x^2-2x+4)}$$

ここで，$\frac{1}{(x+2)(x^2-2x+4)}=\frac{a}{x+2}+\frac{bx+c}{x^2-2x+4}$ とすると，

$$1=a(x^2-2x+4)+(x+2)(bx+c)$$

$$=(a+b)x^2+(-2a+2b+c)x+4a+2c$$

$1=(a+b)x^2+(-2a+2b+c)x+4a+2c$ は恒等式だから，

$$a+b=0,\ -2a+2b+c=0,\ 4a+2c=1 \quad \therefore a=\frac{1}{12},\ b=-\frac{1}{12},\ c=\frac{1}{3}$$

よって，

$$\frac{1}{x^3+8}=\frac{1}{12(x+2)}+\frac{-\frac{1}{12}x+\frac{1}{3}}{x^2-2x+4}$$

$$=\frac{1}{12(x+2)}-\frac{x-4}{12(x^2-2x+4)}$$

$$=\frac{1}{12}\left(\frac{1}{x+2}-\frac{x-4}{x^2-2x+4}\right)$$

$$\int_0^1 \frac{1}{x^3+8}dx = \frac{1}{12}\int_0^1\left(\frac{1}{x+2}-\frac{x-4}{x^2-2x+4}\right)dx$$

$$=\frac{1}{12}\int_0^1\left\{\frac{1}{x+2}-\frac{\frac{1}{2}\left(x^2-2x+4\right)'-3}{x^2-2x+4}\right\}dx$$

$$=\frac{1}{12}\int_0^1\left\{\frac{1}{x+2}-\frac{1}{2}\cdot\frac{\left(x^2-2x+4\right)'}{x^2-2x+4}+\frac{3}{(x-1)^2+3}\right\}dx$$

$$=\frac{1}{12}\int_0^1\left\{\frac{1}{x+2}-\frac{1}{2}\frac{\left(x^2-2x+4\right)'}{x^2-2x+4}\right\}dx+\frac{1}{12}\int_0^1\frac{3}{(x-1)^2+3}dx$$

$$=\frac{1}{12}\left[\log(x+2)-\frac{1}{2}\log\left(x^2-2x+4\right)\right]_0^1+\frac{1}{12}\int_0^1\frac{3}{(x-1)^2+3}dx$$

$$=\frac{\log 3}{24}+\frac{1}{12}\int_0^1\frac{3}{(x-1)^2+3}dx$$

ここで，$\int_0^1\frac{3}{(x-1)^2+3}dx$ について

$x-1=\sqrt{3}\tan\theta$ とおくと，$dx=\frac{\sqrt{3}d\theta}{\cos^2\theta}$，$x=1\Rightarrow\theta=0$，$x=0\Rightarrow\theta=-\frac{\pi}{6}$ より，

$$\int_0^1\frac{3}{(x-1)^2+3}dx=\int_{-\frac{\pi}{6}}^0\frac{3}{3\left(\tan^2\theta+1\right)}\frac{\sqrt{3}d\theta}{\cos^2\theta}=\sqrt{3}\int_{-\frac{\pi}{6}}^0 d\theta=\sqrt{3}[\theta]_{-\frac{\pi}{6}}^0=\frac{\sqrt{3}}{6}\pi$$

よって，

$$\int_0^1\frac{1}{x^3+8}dx=\frac{\log 3}{24}+\frac{1}{12}\int_0^1\frac{3}{(x-1)^2+3}dx$$

$$=\frac{\log 3}{24}+\frac{1}{12}\cdot\frac{\sqrt{3}}{6}\pi$$

$$=\frac{\log 3}{24}+\frac{\sqrt{3}}{72}\pi$$

あるいは，

$$\int_0^1 \frac{1}{x^3+8}dx = \frac{1}{12}\int_0^1\left(\frac{1}{x+2}-\frac{x-4}{x^2-2x+4}\right)dx$$
$$=\frac{1}{12}\int_0^1\left\{\frac{1}{x+2}-\frac{x-4}{(x-1)^2+3}\right\}dx$$
$$=\frac{1}{12}\int_0^1\frac{1}{x-2}dx-\frac{1}{12}\int_0^1\frac{x-4}{(x-1)^2+3}dx$$
$$=\frac{1}{12}\left[\log(x+2)\right]_0^1-\frac{1}{12}\int_0^1\frac{x-4}{(x-1)^2+3}dx$$
$$=\frac{1}{12}\log\frac{3}{2}-\frac{1}{12}\int_0^1\frac{x-4}{(x-1)^2+3}dx$$

ここで，$\int_0^1\frac{x-4}{(x-1)^2+3}dx$ について，

$x-1=\sqrt{3}\tan\theta$ とおくと，$dx=\frac{\sqrt{3}d\theta}{\cos^2\theta}$，$x=1\Rightarrow\theta=0$，$x=0\Rightarrow\theta=-\frac{\pi}{6}$ より，

$$\int_0^1\frac{x-4}{(x-1)^2+3}dx=\int_{-\frac{\pi}{6}}^0\frac{\sqrt{3}\tan\theta-3}{3(\tan^2\theta+1)}\cdot\frac{\sqrt{3}d\theta}{\cos^2\theta}$$
$$=\int_{-\frac{\pi}{6}}^0\left(\tan\theta-\sqrt{3}\right)d\theta$$
$$=\left[-\log(\cos\theta)-\sqrt{3}\theta\right]_{-\frac{\pi}{6}}^0$$
$$=\log\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{6}\pi$$

$$\int_0^1\frac{1}{x^3+8}dx=\frac{1}{12}\log\frac{3}{2}-\frac{1}{12}\int_0^1\frac{x-4}{(x-1)^2+3}dx$$
$$=\frac{1}{12}\log\frac{3}{2}-\frac{1}{12}\left(\log\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{6}\pi\right)$$
$$=\frac{1}{12}\log\sqrt{3}+\frac{\sqrt{3}}{72}\pi$$
$$=\frac{\log 3}{24}+\frac{\sqrt{3}}{72}\pi$$