積分法 7　定積分の部分積分法

$\int_0^{\frac{\pi}{2}} \sin^n x dx$ の値

$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ とおくと，

$$
\begin{aligned}
I_{n+2} &= \int_0^{\frac{\pi}{2}} \sin^{n+2} x dx \\
&= -\int_0^{\frac{\pi}{2}} (\cos x)' \sin^{n+1} x dx \\
&= -\left[\cos x \sin^{n+1} x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \left(\sin^{n+1} x\right)' dx \\
&= 0 + (n+1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^n x dx \\
&= (n+1)\int_0^{\frac{\pi}{2}} \left(1 - \sin^2 x\right)\sin^n x dx \\
&= (n+1)\int_0^{\frac{\pi}{2}} \sin^n x dx - (n+1)\int_0^{\frac{\pi}{2}} \sin^{n+2} x dx \\
&= (n+1)I_n - (n+1)I_{n+2}
\end{aligned}
$$

より，$I_{n+2} = \dfrac{n+1}{n+2} I_n$

また，$I_0 = \int_0^{\frac{\pi}{2}} dx = \dfrac{\pi}{2}$，　$I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$

よって，n が偶数のとき，

$$
\begin{aligned}
I_n &= \frac{n-1}{n} I_{n-2} \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2} I_{n-4} = \cdots \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{3}{4} I_2 \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2} I_0 \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}
\end{aligned}
$$

n が奇数のときも同様にして，

$$
\begin{aligned}
I_n &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{4}{5}\cdot\frac{2}{3} I_1 \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1
\end{aligned}
$$

414

(1)

$$\int_0^1 x\left\{\frac{(x-1)^5}{5}\right\}' dx=\left[\frac{x(x-1)^5}{5}\right]_0^1-\int_0^1\frac{(x-1)^5}{5}dx$$

$$=0-\left[\frac{(x-1)^6}{30}\right]_0^1$$

$$=\frac{1}{30}$$

部分積分を使わない解法

$$\int_0^1\{(x-1)+1\}(x-1)^4dx=\int_0^1\left\{(x-1)^5+(x-1)^4\right\}dx$$

$$=\left[\frac{(x-1)^6}{6}+\frac{(x-1)^5}{5}\right]_0^1$$

$$=\frac{1}{30}$$

(2)

$$\int_0^{\frac{\pi}{2}}(x+2)(\sin x)'dx=[(x+2)\sin x]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin xdx$$

$$=\frac{\pi}{2}+2-[-\cos x]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}+1$$

(3)

$$\int_1^2\left(\frac{x^5}{5}\right)'\log xdx=\left[\frac{x^5}{5}\log x\right]_1^2-\int_1^2\frac{x^4}{5}dx$$

$$=\frac{32}{5}\log 2-\left[\frac{x^5}{25}\right]_1^2$$

$$=\frac{32}{5}\log 2-\frac{31}{25}$$

415

(1)

$$\int_a^b (x-a)^2(x-b)dx = \int_a^b \left\{\frac{(x-a)^3}{3}\right\}'(x-b)dx$$

$$= \left[\frac{(x-a)^3(x-b)}{3}\right]_a^b - \int_a^b \frac{(x-a)^3}{3}dx$$

$$= -\left[\frac{(x-a)^4}{12}\right]_a^b$$

$$= -\frac{1}{12}(b-a)^4$$

部分積分を使わない解法

$$\int_a^b (x-a)^2\{(x-a)-(b-a)\}dx = \int_a^b \left\{(x-a)^3-(b-a)(x-a)^2\right\}dx$$

$$= \left[\frac{(x-a)^4}{4} - \frac{(b-a)(x-a)^3}{3}\right]_a^b$$

$$= -\frac{1}{12}(b-a)^4$$

(2)

$$\int_a^b (x-a)(x-b)^3dx = \int_a^b (x-a)\left\{\frac{(x-b)^4}{4}\right\}'dx$$

$$= \left[\frac{(x-a)(x-b)^4}{4}\right]_a^b - \int_a^b \frac{(x-b)^4}{4}dx$$

$$= -\left[\frac{(x-b)^5}{20}\right]_a^b$$

$$= \frac{1}{20}(a-b)^5$$

部分積分を使わない解法

$$\int_a^b \{(x-b)-(a-b)\}(x-b)^3dx = \int_a^b \left\{(x-b)^4-(a-b)(x-b)^3\right\}dx$$

$$= \left[\frac{(x-b)^5}{5} - \frac{(a-b)(x-b)^4}{4}\right]_a^b$$

$$= \frac{1}{20}(a-b)^5$$

416

(1)

$$\int_1^e \frac{\log x}{x^2}dx = \int_1^e \left(-\frac{1}{x}\right)' \log x dx$$

$$= \left[-\frac{\log x}{x}\right]_1^e + \int_1^e \frac{1}{x}(\log x)' dx$$

$$= -\frac{1}{e} + \int_1^e \frac{1}{x^2}dx$$

$$= -\frac{1}{e} + \left[-\frac{1}{x}\right]_1^e$$

$$= 1 - \frac{2}{e}$$

(2)

$$\int_0^{\frac{\pi}{3}} \frac{x}{\cos^2 x}dx = \int_0^{\frac{\pi}{3}} x(\tan x)' dx$$

$$= [x \tan x]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} x' \tan x dx$$

$$= \frac{\sqrt{3}}{3}\pi - \int_0^{\frac{\pi}{3}} \tan x dx$$

$$= \frac{\sqrt{3}}{3}\pi + [\log(\cos x)]_0^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{3}\pi - \log 2$$

(3)

$$\int_0^1 x^2 e^{2x} dx = \int_0^1 x^2 \left(\frac{e^{2x}}{2}\right)' dx$$

$$= \left[x^2 \cdot \frac{e^{2x}}{2}\right]_0^1 - \int_0^1 (x^2)' \frac{e^{2x}}{2} dx$$

$$= \frac{e^2}{2} - \int_0^1 x e^{2x} dx$$

$$= \frac{e^2}{2} - \int_0^1 x \left(\frac{e^{2x}}{2}\right)' dx$$

$$= \frac{e^2}{2} - \left[x \cdot \frac{e^{2x}}{2}\right]_0^1 + \int_0^1 x' \frac{e^{2x}}{2} dx$$

$$=\frac{e^2}{2}-\frac{e^2}{2}+\left[\frac{e^{2x}}{4}\right]_0^1$$
$$=\frac{e^2-1}{4}$$

417

$$(ax-\sin x)^2=a^2x^2-2ax\sin x+\sin^2 x$$
$$=a^2x^2-2ax\sin x+\frac{1-\cos 2x}{2}$$

より，

$$\int_0^{\frac{\pi}{2}}(ax-\sin x)^2dx=a^2\int_0^{\frac{\pi}{2}}x^2dx-2a\int_0^{\frac{\pi}{2}}x\sin xdx+\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 2x)dx$$
$$=a^2\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}}+2a\int_0^{\frac{\pi}{2}}x(\cos x)'dx+\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$=\frac{\pi^3}{24}a^2+2a[x\cos x]_0^{\frac{\pi}{2}}-2a\int_0^{\frac{\pi}{2}}\cos xdx+\frac{\pi}{4}$$
$$=\frac{\pi^3}{24}a^2+0-2a[\sin x]_0^{\frac{\pi}{2}}+\frac{\pi}{4}$$
$$=\frac{\pi^3}{24}a^2-2a+\frac{\pi}{4}$$
$$=\frac{\pi^3}{24}\left(a-\frac{24}{\pi^3}\right)^2-\frac{24}{\pi^3}+\frac{\pi}{4}$$

よって，$a=\dfrac{24}{\pi^3}$

418

解法 1：部分積分 1

$$\int e^{-3x}\sin xdx=-\int e^{-3x}(\cos x)'dx$$
$$=-e^{-3x}\cos x+\int\left(e^{-3x}\right)'\cos xdx$$
$$=-e^{-3x}\cos x-3\int e^{-3x}\cos xdx$$
$$=-e^{-3x}\cos x-3\int e^{-3x}(\sin x)'dx$$
$$=-e^{-3x}\cos x-3\left\{e^{-3x}\sin x-\int\left(e^{-3x}\right)'\sin xdx\right\}$$
$$=-e^{-3x}\cos x-3\left(e^{-3x}\sin x+3\int e^{-3x}\sin xdx\right)$$
$$=-e^{-3x}(\cos x+3\sin x)-9\int e^{-3x}\sin xdx$$

より，$\int e^{-3x}\sin xdx=-\frac{1}{10}e^{-3x}(\cos x+3\sin x)+C$

よって，

$$\int_0^{\frac{\pi}{2}} e^{-3x}\sin xdx=-\frac{1}{10}\left[e^{-3x}(\cos x+3\sin x)\right]_0^{\frac{\pi}{2}}$$
$$=-\frac{1}{10}\left(3e^{-\frac{3}{2}\pi}-1\right)$$
$$=\frac{1}{10}\left(1-3e^{-\frac{3}{2}\pi}\right)$$

解法 2：部分積分 2

$$\int e^{-3x}\sin xdx=-\int\left(\frac{1}{3}e^{-3x}\right)'\sin xdx$$
$$=-\frac{1}{3}e^{-3x}\sin x+\frac{1}{3}\int e^{-3x}(\sin x)'dx$$
$$=-\frac{1}{3}e^{-3x}\sin x+\frac{1}{3}\int e^{-3x}\cos xdx$$
$$=-\frac{1}{3}e^{-3x}\sin x-\frac{1}{3}\int\left(\frac{1}{3}e^{-3x}\right)'\cos xdx$$
$$=-\frac{1}{3}e^{-3x}\sin x-\frac{1}{3}\left\{\frac{1}{3}e^{-3x}\cos x-\frac{1}{3}\int e^{-3x}(\cos x)'dx\right\}$$
$$=\frac{1}{3}e^{-3x}\sin x-\frac{1}{3}\left(\frac{1}{3}e^{-3x}\cos x+\frac{1}{3}\int e^{-3x}\sin xdx\right)$$
$$=-\frac{1}{9}e^{-3x}(3\sin x+\cos x)-\frac{1}{9}\int e^{-3x}\sin xdx$$

より，$\int e^{-3x}\sin xdx=-\frac{1}{10}e^{-3x}(\cos x+3\sin x)+C$

よって，

$$\int_0^{\frac{\pi}{2}} e^{-3x}\sin xdx=-\frac{1}{10}\left[e^{-3x}(\cos x+3\sin x)\right]_0^{\frac{\pi}{2}}$$
$$=-\frac{1}{10}\left(3e^{-\frac{3}{2}\pi}-1\right)$$
$$=\frac{1}{10}\left(1-3e^{-\frac{3}{2}\pi}\right)$$

解法 3：連立方程式

$$\left(e^{-3x}\sin x\right)' = -3e^{-3x}\sin x + e^{-3x}\cos x \quad \cdot\cdot\cdot ①$$

$$\left(e^{-3x}\cos x\right)' = -e^{-3x}\sin x - 3e^{-3x}\cos x \quad \cdot\cdot\cdot ②$$

3×①＋②より，

$$3\left(e^{-3x}\sin x\right)' + \left(e^{-3x}\cos x\right)' = -10e^{-3x}\sin x$$

よって， $e^{-3x}\sin x = -\frac{1}{10}\left\{e^{-3x}(3\sin x + \cos x)\right\}'$

ゆえに，

$$\int_0^{\frac{\pi}{2}} e^{-3x}\sin x dx = -\frac{1}{10}\left[e^{-3x}(3\sin x + \cos x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{10}\left(1 - 3e^{-\frac{3}{2}\pi}\right)$$

419

(1)

$$I_1 = \int_1^e \log x dx$$
$$= \left[x(\log x - 1)\right]_1^e$$
$$= 1$$

(2)

$$I_{n+1} = \int_1^e (\log x)^{n+1} dx$$
$$= \int_1^e x'(\log x)^{n+1} dx$$
$$= \left[x(\log x)^{n+1}\right]_1^e - \int_1^e x\left\{(\log x)^{n+1}\right\}' dx$$
$$= e - (n+1)\int_1^e (\log x)^n dx$$
$$= e - (n+1)I_n$$

(3)

$$I_2 = e - 2I_1 = e - 2$$
$$I_3 = e - 3I_2 = e - 3(e-2) = -2e + 6$$
$$\therefore I_4 = e - 4I_3 = e - 4(-2e + 6) = 9e - 24$$