積分法の応用 3　曲線の長さ

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(1)

$$L=\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$=\int_0^1\sqrt{\left(2t^2\right)^2+(2t)^2}\,dt$$
$$=\int_0^1 2t\left(t^2+1\right)^{\frac{1}{2}}dt$$
$$=\left[\frac{2}{3}\left(t^2+1\right)^{\frac{3}{2}}\right]_0^1$$
$$=\frac{2}{3}\left(2\sqrt{2}-1\right)$$

(2)

$$L=\int_0^{\sqrt{3}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$=\int_0^{\sqrt{3}}\sqrt{(6t)^2+\left(3-3t^2\right)^2}\,dt$$
$$=3\int_0^{\sqrt{3}}\left(t^2+1\right)dt$$
$$=3\left[\frac{t^3}{3}+t\right]_0^{\sqrt{3}}$$
$$=6\sqrt{3}$$

(3)

$$L=\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$
$$=\int_0^{\pi}\sqrt{\left\{e^{\theta}(\cos\theta-\sin\theta)\right\}^2+\left\{e^{\theta}(\sin\theta+\cos\theta)\right\}^2}\,d\theta$$
$$=\sqrt{2}\int_0^{\pi}e^{\theta}dt$$
$$=\sqrt{2}\left[e^{\theta}\right]_0^{\pi}$$
$$=\sqrt{2}\left(e^{\pi}-1\right)$$

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(1)

$$L=\int_2^3\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\int_2^3\sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}\,dx$$
$$=\int_2^3\left(x^2+\frac{1}{4x^2}\right)dx$$
$$=\left[\frac{x^3}{3}-\frac{1}{4x}\right]_2^3$$
$$=\frac{51}{8}$$

(2)

$$L=\int_{-1}^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\int_{-1}^1\sqrt{1+\left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\,dx$$
$$=\int_{-1}^1\frac{2}{\sqrt{4-x^2}}\,dx$$

ここで，$x=2\sin\theta\left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$とおくと，

$dx=2\cos\theta d\theta$，$x=1\Rightarrow\theta=\frac{\pi}{6}$，$x=-1\Rightarrow\theta=-\frac{\pi}{6}$より，

$$L=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{2}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta d\theta$$
$$=2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta$$
$$=2\left[\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$
$$=\frac{2}{3}\pi$$

(3)

$$L=\int_0^{\frac{1}{2}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\int_0^{\frac{1}{2}}\sqrt{1+\left(-\frac{2x}{1-x^2}\right)^2}\,dx$$
$$=\int_0^{\frac{1}{2}}\left|\frac{1+x^2}{1-x^2}\right|dx$$
$$=\int_0^{\frac{1}{2}}\frac{1+x^2}{1-x^2}\,dx \quad \left(\because 0\le x\le\frac{1}{2}\Rightarrow\frac{1+x^2}{1-x^2}>0\right)$$
$$=\int_0^{\frac{1}{2}}\left(-1+\frac{2}{1-x^2}\right)dx$$
$$=\int_0^{\frac{1}{2}}\left\{-1+\frac{2}{(1-x)(1+x)}\right\}dx$$
$$=\int_0^{\frac{1}{2}}\left(-1+\frac{1}{1-x}+\frac{1}{1+x}\right)dx$$
$$=\left[-x-\log(1-x)+\log(1+x)\right]_0^{\frac{1}{2}}$$
$$=-\frac{1}{2}+\log 3$$

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(1)

$$L=\int_0^{\alpha}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$
$$=\int_0^{\alpha}\sqrt{(a\theta\cos\theta)^2+(a\theta\sin\theta)^2}\,dx$$
$$=\int_0^{\alpha}|a\theta|dx$$
$$=a\int_0^{\alpha}\theta dx \quad (\because 0\le\theta\le\alpha,\ a>0\Rightarrow a\theta\ge 0)$$
$$=a\left[\frac{\theta^2}{2}\right]_0^{\alpha}$$
$$=\frac{a\alpha^2}{2}$$

(2)

$$
\begin{aligned}
L &= \int_0^{\beta} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_0^{\beta} \sqrt{1+\left(-\frac{\sin x}{\cos x}\right)^2}\,dx \\
&= \int_0^{\beta} \sqrt{1+\tan^2 x}\,dx \\
&= \int_0^{\beta} \sqrt{\frac{1}{\cos^2 x}}\,dx \\
&= \int_0^{\beta} \frac{1}{|\cos x|}\,dx \\
&= \int_0^{\beta} \frac{1}{\cos x}\,dx \quad \left(\because 0 \le x \le \beta < \frac{\pi}{2} \Rightarrow \cos x > 0\right) \\
&= \int_0^{\beta} \frac{\cos x}{\cos^2 x}\,dx \\
&= \int_0^{\beta} \frac{\cos x}{1-\sin^2 x}\,dx \\
&= \int_0^{\beta} \frac{\cos x}{(1-\sin x)(1+\sin x)}\,dx \\
&= \frac{1}{2}\int_0^{\beta} \cos x\left(\frac{1}{1-\sin x}+\frac{1}{1+\sin x}\right)dx \\
&= \frac{1}{2}\int_0^{\beta} \left(\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}\right)dx \\
&= \frac{1}{2}\Big[-\log(1-\sin x)+\log(1+\sin x)\Big]_0^{\beta} \\
&= \frac{1}{2}\log\frac{1+\sin\beta}{1-\sin\beta}
\end{aligned}
$$

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曲線 $x=\cos\theta,\ y\sin\theta\ \left(0\leq\theta\leq 2\pi\right)$ の長さは,

$$\int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_0^{2\pi}\sqrt{(\cos\theta-\theta\sin\theta)^2+(\sin\theta+\theta\cos\theta)^2}\,d\theta$$

$$=\int_0^{2\pi}\sqrt{1+\theta^2}\,d\theta$$

曲線 $y=\dfrac{1}{2}x^2\ \left(0\leq x\leq 2\pi\right)$ の長さは,

$$\int_0^{2\pi}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{2\pi}\sqrt{1+x^2}\,dx$$

ここで, $x=\theta$ とおくと, $dx=d\theta$, $x=2\pi\Rightarrow\theta=2\pi,\ x=0\Rightarrow\theta=0$ より,

$$\int_0^{2\pi}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{2\pi}\sqrt{1+\theta}\,d\theta$$

よって,

曲線 $x=\cos\theta,\ y\sin\theta\ \left(0\leq\theta\leq 2\pi\right)$ の長さは, 曲線 $y=\dfrac{1}{2}x^2\ \left(0\leq x\leq 2\pi\right)$ の長さに等しい。